Gauss' Law

If you go on in Physics you will learn all about GAUSS' LAW along with vector calculus in your advanced course on ELECTRICITY AND MAGNETISM, where it is used to calculate the electric field strength at various distances from highly symmetric distributions of electric charge. However, GAUSS' LAW can be applied to a huge variety of interesting situations having nothing to do with electricity except by analogy. Moreover, the rigourous statement of GAUSS' LAW in the mathematical language of vector calculus is not the only way to express this handy concept, which is one of the few powerful modern mathematical tools which can be accurately deduced from "common sense" and which really follows from a statement so simple and obvious as to seem trivial and uninteresting, to wit:

(Colloquial form of GAUSS' LAW)

"When something passes out of a region, it is no longer inside that region."

How, you may ask, can such a dumb tautology teach us anything we don't already know? The power of GAUSS' LAW rests in its combination with our knowledge of *geometry* (*e.g.* the surface area A of a sphere of radius r is $A = 4\pi r^2$) and our instinctive understanding of *symmetry* (*e.g.* there is no way for a *point* of zero size to define a favoured *direction*). When we put these two skills together with GAUSS' LAW we are able to easily derive some not-so-obvious *quantitative* properties of many commonly-occurring natural phenomena.

18.1 The Point Source

For example, consider a hypothetical "spherically symmetric" *sprinkler head* (perhaps meant to uniformly irrigate the inside surface of a hollow spherical space colony): located at the centre of the sphere, it "emits" (squirts out) dQ/dt gallons per second of water *in all directions equally*, which is what we mean by "spherically symmetric" or "*isotropic*."[1] Here Q is the "amount of stuff" — in this case measured in gallons. Obviously (beware of that word, but it's OK here), since water is *conserved* the total *flow* of water is conserved: once a "steady-state" (equilibrated) flow has been established, the rate at which water is deposited on the walls of the sphere is the same as the rate at which water is emitted from the sprinkler head at the centre. That is, if we add up (integrate) the "flux" $\vec{\boldsymbol{J}}$ of water per second per square meter of surface area at the sphere wall over the whole spherical surface, we must get dQ/dt. Mathematically, this is written

$$\oint\!\!\!\oint_{\mathcal{S}} \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{A}} = \frac{dQ}{dt} \tag{1}$$

where the $\oint\!\!\oint_{\mathcal{S}}$ stands for an integral (sum of elements) over a closed surface $\mathcal{S}$. [This part is crucial, inasmuch as an open surface (like a hemisphere) does not account for all the flux and cannot be used with GAUSS' LAW]. Now, we must pay a little attention to the *vector* notation: the "flux" $\vec{\boldsymbol{J}}$ always has a *direction*, like the flux (current) of water flowing in a river or in this case the flux of water droplets passing through space.

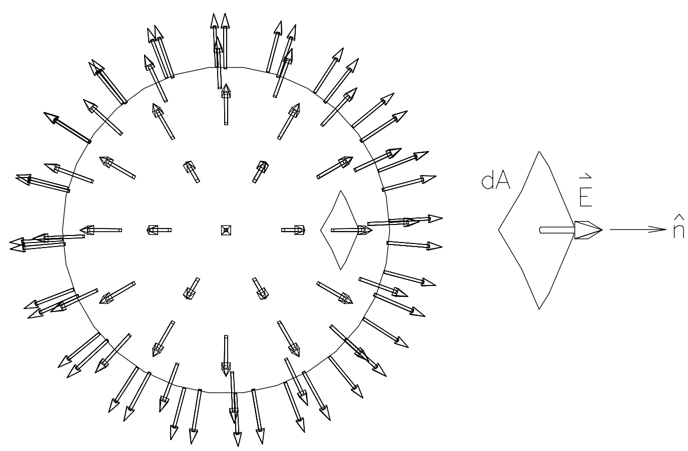

Figure 18.1 An isotropic source.

Each droplet has a (vector) velocity, and the velocity and the density of droplets combine to form the "flux" as described above. Not so trivial is the idea of a vector *area* element $d\vec{\boldsymbol{A}}$, but the sense of this is clear if we think of what happens to the scalar flux (in gallons/sec) through a hoop of wire of area $d\vec{\boldsymbol{A}}$ when we place it in a river: if the direction of the flow of the river is perpendicular ("normal") to the plane of the hoop, we get the maximum possible flux, namely the vector flux magnitude (the flow rate of the river) times the area of the hoop; if we reorient the hoop so that its area intercepts no flow (*i.e.* if the direction $\hat{n}$ "normal" to the plane of the hoop is perpendicular to the direction of flow of the river) then we get zero flux through the hoop. In general, the scalar rate of flow (here measured in gallons/sec) through a "surface element" $d\vec{\boldsymbol{A}}$ whose "normal" direction $\hat{n}$ is given by $(\vec{\boldsymbol{J}} \cdot \hat{n})dA$ or just $\vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{A}}$

[1] Note how our terminology of spherical coordinates stems from terrestrial navigation (Tropics of Cancer, Capricorn, *etc.*). Since the 16^{th} Century, our most familiar spherical object (next to the cannonball) has been the Earth.

where we have now defined the *vector* surface element $d\vec{\boldsymbol{A}} \equiv \hat{n} dA$. This is pictured in Fig. 18.1 above.

Returning now to our sprinkler-head example, we have a Law [Eq. (1)] which is a mathematical (and therefore quantitative) statement of the colloquial form, which in principle allows us to calculate something. However, it is still of only academic interest in general. Why? Because the integral described in Eq. (1) is so general that it may well be hopelessly difficult to solve, unless (!) there is something about the *symmetry* of the particular case under consideration that makes it easy, or even "trivial." Fortunately (though hardly by accident) in this case there is — namely, the isotropic nature of the sprinkler head's emission, plus the spherically symmetric (in fact, spherical) shape of the surface designated by "$\mathcal{S}$" in Eq. (1). These two features ensure that

1. the *magnitude* $J = |\vec{\boldsymbol{J}}|$ of the flux is the same everywhere on the surface $\mathcal{S}$; and
2. the *direction* of $\vec{\boldsymbol{J}}$ is *normal* to the surface everywhere it hits on $\mathcal{S}$.

In this case, $\vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{A}} = J dA$ and J is now a constant which can be taken outside the integral sign, leaving

$$J \oiint_{\mathcal{S}} dA = \dot{Q}$$

where $\dot{Q}$ is just a compact notation for dQ/dt. But $\oiint_{\mathcal{S}} dA$ is just the *area* of the sphere, $4\pi r^2$, where r is the *radius* of the sphere, so (1) becomes

$$4\pi r^2 J = \dot{Q}$$

or

$$J(r) = \frac{\dot{Q}}{4\pi r^2} \tag{2}$$

which states the *general* conclusion for *any* spherically symmetric emission of a conserved quantity, namely

> *The flux from an isotropic source points away from the centre and falls off proportional to the inverse square of the distance from the source.*

This holds in an amazing variety of situations. For instance, consider the "electric field lines" from a spherically symmetric electric charge distribution as measured at some point a distance r away from the centre. We visualize these electric field "lines" as streams of some mysterious "stuff" being "squirted out" by positive charges (or "sucked in" by negative charges). The idea of an electric field line is of course a pure construct; no one has ever seen or ever will see a "line" of the electric field $\vec{\boldsymbol{E}}$, but if we think of the *strength* of $\vec{\boldsymbol{E}}$ as the "number of field lines per unit area perpendicular to $\vec{\boldsymbol{E}}$" and treat these "lines of force" as if they were *conserved* in the same way as streams of water, we get a useful graphical picture as well as a model which, when translated into mathematics, gives correct answers. As suspicious as this may sound, it is really all one can ask of a physical model of something we cannot see. This is the sense of all sketches showing electric field lines. For every little bit ("element") of charge dq on one side of the symmetric distribution there is an equal charge element exactly opposite (relative to the radius vector joining the centre to the point at which we are evaluating $\vec{\boldsymbol{E}}$); the "transverse" contributions of such charge elements to $\vec{\boldsymbol{E}}$ all cancel out, and so the only possible direction for $\vec{\boldsymbol{E}}$ to point is along the radius vector — *i.e.* as described above. An even simpler argument is that *there is no way to pick a preferred direction* (other than the radial direction) if the charge distribution truly has spherical symmetry. This "symmetry argument" is implied in Fig. 18.1.

Now we must change our notation slightly from the general description of Eqs. (1) and (2) to the specific example of electric charge and field. Inasmuch as one's choice of a system of *units* in electromagnetism is rather flexible, and since each choice introduces a different set of constants of proportionality with odd units of their own, I will merely state that "J turns into E, $dQ/dt \to q$ now stands for *electric charge*, and there is a $1/\epsilon_0$ in front of the $dQ/dt \equiv q$ on the right-hand side of Eq. (1)" to give us the electrostatics version of (1):

$$\oiint_{\mathcal{S}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{q}{\epsilon_0} \tag{3}$$

which, when applied to the isotropic charge distribution, gives the result

$$E(r) = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r^2} \tag{4}$$

The implication of Eq. (3) is then that, since the spherical shell contains the same amount of charge for all radii $r > R$, where R is the physical radius of the charge distribution itself, it cannot matter *how* the charge is distributed (as long as it is spherically symmetric); to the distant observer, the $\vec{\boldsymbol{E}}$ field it produces will always look just like the $\vec{\boldsymbol{E}}$ field due to a *point* charge q at the centre; *i.e.* Eq. (4).

18.1.1 Gravity

Another example is *gravity*, which differs from the electrostatic force only in its relative weakness and the innocuous-looking fact that it only comes in one sign, namely attractive, whereas the electric force can be either attractive (for unlike charges) or repulsive (for like charges). That is, "There are no negative masses." So all these equations hold equally well for gravity, except of course that we must again shuffle constants of proportionality around to make sure we are not setting apples equal to oranges. In this case we can use some symbol, say $\vec{g}$, to represent the force *per unit mass* at some position, as we did for $\vec{E}$ = force per unit *charge*, and talk about the "gravitational field" as if it were really there, rather than being what would be there (a force) if we placed a mass there. (Note that $\vec{g}$ will be measured in units of acceleration.) Then the role of "dQ/dt" in Eq. (1) is played by M, the total mass of the attracting body, and the constant of proportionality is $4\pi G$, where G is Newton's Universal Gravitational Constant:

$$\oint\!\!\oint_{S} \vec{g} \cdot d\vec{A} = 4\pi G M \tag{5}$$

and

$$g(r) = \frac{GM}{r^2} \tag{6}$$

for any spherically symmetric mass distribution of total mass M. Note that we have "derived" this fundamental relationship from arguments about symmetry, geometry and common sense, plus the weird notion that "lines" of gravitational force are "emitted" by masses and are "conserved" in the sense of streams of water — a pretty kinky idea, but evidently one with powerful applications. Be sure you are satisfied that this is *not* a "circular argument;" we really have derived Eq. (6) without using it in the development at all! Now, besides being suggestive of deeper knowledge, this trick can be used to draw amusing conclusions about interesting physical situations.

The Spherical Shell

For instance, suppose that one day we assemble all the matter in the Solar System and build one gigantic spherical shell out of it. We arrange its radius so that the force of gravity at its surface (standing on the outside) is "Earth normal," *i.e.* 9.81 N/kg or $g = 9.81$ m/s^2. This is all simple so far, and GAUSS' LAW tells us that as long as we are *outside* of the spherical shell enclosing the whole spherically symmetric mass distribution, the gravitational field we will see is indistinguishable from that produced by the entire mass concentrated at a point at the centre. The amazing prediction is that if we merely step *inside* the shell, there is still spherical symmetry, but the spherical surface touching our new radius *does not enclose any mass* and therefore sees *no* gravitational field at all! This is actually correct: inside the sphere we are weightless, and travel opportunities to other parts of the shell (across the inside) become quite interesting.

There are many more examples of entertaining properties of spherically symmetric charge or mass distributions, all of which you can easily deduce from similar arguments to dazzle your friends. Let us now ask, however, if any *less symmetric* situations can also be treated easily with this technique.

18.1.2 The Uniform Sphere

Another familiar example of spherical symmetry is the uniformly dense solid sphere of mass (if we are interested in gravity) or the solid sphere of insulating material carrying a uniform charge density ρ (if we want to do electrostatics). Let's pick the latter, just for variety. If we imagine a spherical "Gaussian surface" concentric with the sphere, with a radius r less than the sphere's radius R, the usual isotropic symmetry argument gives $\oint\!\!\oint_S \vec{E} \cdot d\vec{A} = 4\pi r^2 E$, where E is the (constant) radial electric field strength at radius $r < R$. The net charge enclosed within the Gaussian surface is $\frac{4}{3}\pi r^3 \rho$, giving $4\pi r^2 E = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho$, or

$$E(r < R) = \frac{\rho}{3\epsilon_0} r \tag{7}$$

for the electric field inside such a uniform spherical charge density.

A similar linear relationship holds for the gravitational field within a solid sphere of uniform mass density, of course, except in that case the force on a "test mass" is always back toward the centre of the sphere — *i.e.* a *linear restoring force* with all that implies....

18.2 The Line Source

A sphere, as we have seen, can be collapsed to a point without affecting the external field; and a point is essentially a "zero-dimensional object" — it has no properties that can help us to define a unique direction in space. The next higher-dimensional object would be *one*-dimensional, namely a *line*. What can we do with this?

In the spirit of the normal physics curriculum, we will now stick to the example of electrostatics, remembering that all the same arguments can be used on gravity or indeed on other situations not involving "force fields" at all. (Consider the sprinkler, or a source of "rays" of light.) Suppose that we have an "infinite line of charge," *i.e.* a straight wire with a charge λ *per unit length*. This is pictured in Fig. 18.2.

The same sort of symmetry arguments used in Fig. 18.1 tell us that for every element of charge a distance d

above position x on the wire, there is an equal element of charge an equal distance d *below* position x, from which we can conclude that the "transverse" contributions to the $\vec{E}$ field from the opposite charge elements cancel, leaving only the components pointing *directly away from the wire*; *i.e.* perpendicular to the wire. In what are referred to as "cylindrical coordinates," the perpendicular distance from the wire to our field point is called r, and the direction described above is the r direction. Thus $\vec{E}$ points in the $\hat{r}$ direction. (Indeed, if it wanted to point in another direction, it would have to choose it arbitrarily, as there *is* no other direction that can be defined uniquely by reference to the wire's geometry!) Given the direction of $\vec{E}$ and the "obvious" (but nevertheless correct) fact that it must have the same *strength* in all directions (*i.e.* it must be independent of the "azimuthal angle" ϕ — another descriptive term borrowed from celestial navigation), we can guess at a shape for the closed surface of Eq. (3) which will give us $\vec{E}$ either parallel to the surface (no contribution to the outgoing flux) or normal to the surface and constant, which will let us take E outside the integral and just determine the total area perpendicular to $\vec{E}$: we choose a cylindrical shaped "pillbox" centred on the wire. No flux escapes from the "end caps" because $\vec{E}$ is parallel to the surface; $\vec{E}$ is constant in magnitude over the curved outside surface and everywhere perpendicular (normal) to it. Thus

$$\oiint_S \vec{E} \cdot d\vec{A} = E \oiint_S dA = (E)(2\pi r L)$$

where $2\pi r L$ is the curved surface area of a cylinder of radius r and height L.

The same surface, clipping off a length L of wire, encloses a net charge $q = \lambda L$. Plugged into (3), this gives

$$2\pi r L E = \frac{\lambda L}{\epsilon_0}$$

or

$$E(r) = \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{1}{r} \qquad (8)$$

which states the *general* conclusion for *any cylindrically* symmetric charge distribution, namely that

> *The electric field from a cylindrically symmetric charge distribution points away from the central line and falls off proportional to the inverse of the distance from the centre.*

This also holds in an amazing variety of situations. Applications are left to the interested student.

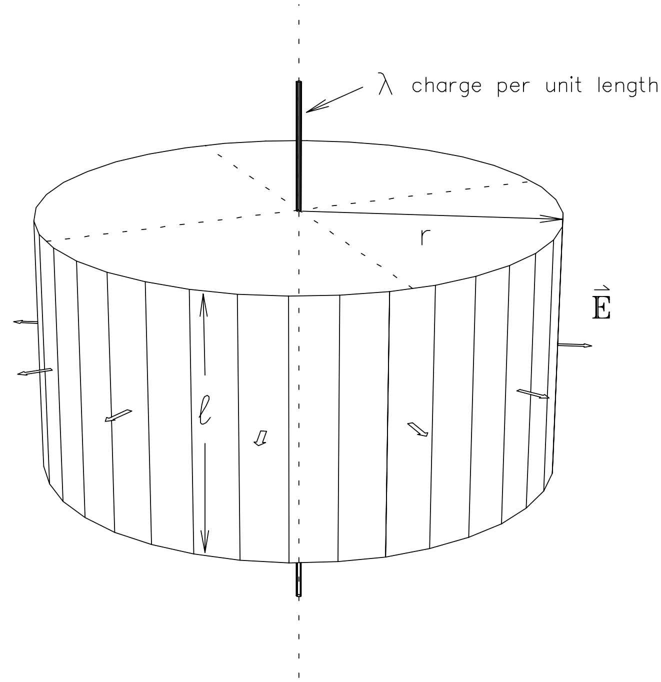

Figure 18.2 An infinite, uniform line of charge.

18.3 The Plane Source

Note the interesting trend: a zero-dimensional distribution (a point) produces a field that drops off as r^{-2}, while a one-dimensional distribution (a line) produces a field that drops off as r^{-1}. We have to be tempted to see if a *two*-dimensional distribution (a *plane*) will give us a field that drops off as r^0 — *i.e.* which does not drop off at all with the distance from the plane, but remains *constant* throughout space. This application of GAUSS' LAW is a straightforward analogy to the other two, and can be worked out easily by the reader. ;-)