The University of British Columbia

# Physics 108 Assignment # 7 SOLUTIONS:

# THE MAGNETIC FIELD

Wed. 23 Feb. 2005 — finish by Wed. 2 Mar.

**1.** **FRICTION** *vs.* **THE LORENTZ FORCE:** A 2-kg copper rod rests on two horizontal rails 2 m apart and carries a current of 100 A from one rail to the other. The coefficient of static friction between the rod and the rails is $\mu_s = 0.5$. What is the smallest magnetic field (not necessarily vertical) that would cause the bar to slide?
**ANSWER**: First let's calculate the minimum required magnetic force: the usual FBD gives $N = mg - F\sin\theta$ and $F\cos\theta = \mu N$ just before slipping. Solving for $F$ yields $F = \mu mg/[\cos\theta + \mu\sin\theta]$. Differentiating with respect to $\theta$ gives $dF/d\theta = \mu mg(\sin\theta - \mu\cos\theta)/(\cos\theta + \mu\sin\theta)^2$, which is zero (the condition for an extremum) when $\sin\theta = \mu\cos\theta$ or when $\tan\theta = \mu = 0.5 \Longrightarrow \theta = 0.4636$. Thus $F = (0.5 \times 2 \times 9.81)/(0.8944 + 0.5 \times 0.4472) = 8.774$ N. Since the current is out of the page, we want a magnetic field $\vec{B}$ in the plane of the page at right angles to the desired force, *i.e.* making an angle $\frac{\pi}{2} + \theta = 2.034$ with the horizontal $x$ direction. The magnitude of $B_{\rm min}$ is given by $F = ILB_{\rm min}$ with $L = 2$ m, namely $B_{\rm min} = F/IL = 8.774/(100 \times 2)$ or $\boxed{B_{\rm min} = 0.04387 \text{ T}}$ .

**2.** **CYCLOTRONS:** (Neglect any relativistic effects.) Suppose that we want to build a small cyclotron for protons using a magnet with a uniform field over a region 1 m in radius such that the protons reach a maximum kinetic energy of 20 MeV at the outer radius of the magnet.

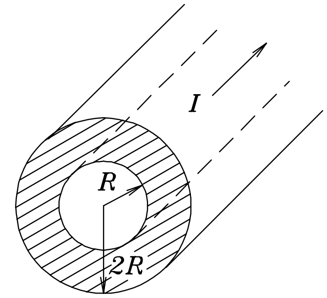
($a$) What magnetic field must the magnet produce? **ANSWER**: In each case we have $m\frac{v^2}{r} = qvB \Longrightarrow mv \equiv p = qBr$ and $\omega = \frac{v}{r} = \frac{qB}{m}$. In this case $q = e = 1.602 \times 10^{-19}$ C and $m = m_p = 1.67 \times 10^{-27}$ kg. At $K = 20$ MeV $= 20 \times 10^6 \times 1.602 \times 10^{-19} = 3.204 \times 10^{-12}$ J, $p = \sqrt{2mK} = 1.035 \times 10^{-19}$ kg-m/s, so when $r = 1$ m, $B = p/qr$ gives $\boxed{B = 0.646 \text{ T}}$ .

($b$) At what frequency must the "dee" voltage oscillate? **ANSWER**: $\omega = qB/m = 6.19 \times 10^7$ s$^{-1}$ and $\boxed{f = \omega/2\pi = 9.85 \text{ MHz}}$ .

Now suppose we want to build a cyclotron to accelerate electrons *without a magnet*, using the Earth's magnetic field (assume $B = 5 \times 10^{-5}$ T) to keep the electrons moving in circles.

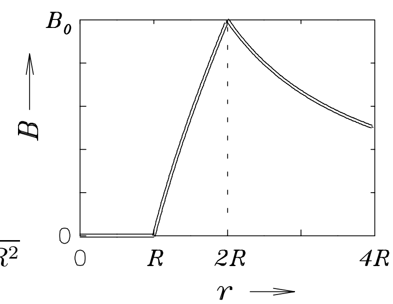
($c$) What is the radius of the electron orbit at 100 eV? **ANSWER**: Now $m = m_e = 9.11 \times 10^{-31}$ kg, $|q| = e$, $B = 5 \times 10^{-5}$ T and $K = 100$ eV $= 100 \times 1.602 \times 10^{-19} = 1.602 \times 10^{-17}$ J $\Longrightarrow p = \sqrt{2mK} = 5.403 \times 10^{-24}$ kg-m/s giving $\boxed{r = p/qB = 0.6744 \text{ m}}$ .

($d$) What is the frequency (in Hz) of the RF electric field we must supply to the cyclotron "dees?"
**ANSWER**: $\omega = qB/m = 8.794 \times 10^6$ s$^{-1}$ and $\boxed{f = \omega/2\pi = 1.40 \times 10^6 \text{ Hz}}$ .




**3. HOLLOW CYLINDRICAL CONDUCTOR:** A thick-walled hollow conducting cylinder carries a uniformly distributed current $I$. The (centred) hole in the middle has a radius of $R$ and the outer radius of the conductor is $2R$. Derive an expression for the strength of the magnetic field $B$ as a function of radial distance $r$ from the cylinder axis, in the range from $r = R$ to $r = 2R$; then *plot* (*i.e.* sketch, showing axis labels, scales and values at key points) $B(r)$ in the range from $r = 0$ to $r = 4R$.

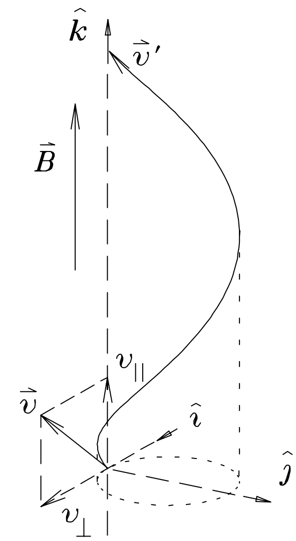
**ANSWER**: Throughout this problem, *symmetry* demands that the magnetic field circulate around the central axis according to the right hand rule and have the same magnitude $B(r)$ everywhere on an Ampèrian loop of radius $r$ centred on the axis. Thus $\oint \vec{\boldsymbol{B}} \cdot d\vec{s} = 2\pi r B = \mu_0 I_{\text{encl}}$ for every such loop and all there is to the calculation is to determine the current $I_{\text{encl}}$ enclosed by the loop for each $r$. When $r < R$, *no* current is enclosed so $B(r < R) = 0$. For $r > 2R$, *all* the current is enclosed so $B(r > 2R) = \dfrac{\mu_0 I}{2\pi r}$ just as for a *wire* along the axis. The only nontrivial case is for $R < r < 2R$, where the enclosed current is given by $\dfrac{I_{\text{encl}}}{I} = \dfrac{\pi r^2 - \pi R^2}{\pi (2R)^2 - \pi R^2}$ — *i.e.* the fraction of current is the fraction of cross-sectional area.




This gives $\boxed{B(R < r < 2R) = \dfrac{\mu_0 I}{2\pi r} \cdot \dfrac{(r^2 - R^2)}{3R^2}}$. At $r = 2R$, $B(2R) \equiv B_0 = \dfrac{\mu_0 I}{4\pi R}$. (See sketch.)

# Challenge Problems:

**1. MOTION OF AN ELECTRON IN A MAGNETIC FIELD:** An electron has a kinetic energy of 400 eV as it moves through a region containing a uniform magnetic field $\vec{B} = B\hat{k}$ of magnitude $B = 4 \times 10^{-4}$ T. At $t = 0$ it is at the origin of coordinates $(x = 0, y = 0, z = 0)$ and has velocity components $v_y = 0$ and $v_x = v_z > 0$. Find the position of the electron ($x$, $y$ and $z$) 10 ns later. [1 ns = $10^{-9}$ s]

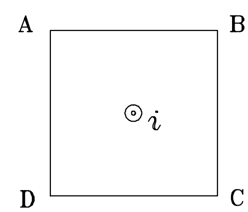



**ANSWER**: The kinetic energy is $K = \frac{1}{2}mv^2 = 400 \times 1.602 \times 10^{-19} = 6.41 \times 10^{-17}$ J and $m = 9.11 \times 10^{-31}$ kg, so $v = \sqrt{2K/m} = 1.186 \times 10^7$ m/s. Since $\vec{v} = v_x\hat{\imath} + v_z\hat{k}$ and $v_x \equiv v_\perp = v_z \equiv v_\|$, we have $v_\| = v_\perp = v/\sqrt{2} = 0.8388 \times 10^7$ m/s. Now, $\vec{B} = B\hat{k}$ so $v_\|$ is parallel to $\vec{B}$ and therefore continues unaffected by the magnetic field, while $v_\perp$ bends in a circle of radius $r = mv_\perp/qB = 0.1192$ m. The electron is negatively charged, so it bends to the left in an "up" field (you can verify this from the right-hand rule for $\vec{F} = q\vec{v} \times \vec{B}$) and so the path is a spiral with constant radius and pitch, as shown in the figure. The frequency of the orbit of $v_\perp$ is given by $\omega = v_\perp/r = qB/m = 0.7035 \times 10^8$ s$^{-1}$. At $t = 10 \times 10^{-9}$ s the angle swept out by the spiral since $t = 0$ is $\theta = \omega t = 0.7035$ radians. This gives $x = r\sin\theta$ and $y = r(1 - \cos\theta)$ the values $\boxed{x = 0.0771 \text{ m}}$ and $\boxed{y = 0.0283 \text{ m}}$. Meanwhile $\boxed{z = v_\| t = 0.0839 \text{ m}}$.

**2. FORCE ON A CURRENT-CARRYING CONDUCTOR:** A long, rigid conductor, lying along the $x$ axis, carries a current of 6 A in the $-\hat{\imath}$ direction. A magnetic field $\vec{B} = 3.0\hat{\imath} + 6x^2\hat{\jmath}$ (with $x$ in m and $B$ in mT) is present. Calculate the vector force on the 3-m segment of the conductor that lies between $x = 1.0$ m and $x = 4$ m.

**ANSWER**: In general $d\vec{F} = Id\vec{\ell} \times \vec{B}$. In this case (suppressing the $SI$ units) we have $Id\vec{\ell} = -6\hat{\imath}dx$ so $d\vec{F} = dx(-6\hat{\imath}) \times (3.0\hat{\imath} + 6x^2\hat{\jmath}) \times 10^{-3} = dx(-0.036x^2)\hat{k}$ (remember, $\hat{\imath} \times \hat{\imath} = 0$) which we must then integrate over the specified range to get $\vec{F} = -0.036\hat{k}\int_{1.0}^{4} x^2\,dx = -\left(\frac{0.036}{3}\right)\hat{k}\left[x^3\right]_{1.0}^{4} = -0.012\hat{k}(64 - 1)$ or $\boxed{\vec{F} = -(0.756 \text{ N})\hat{k}}$.

**3. AMPÈRE'S LAW:** A wire carrying a current of 2001 A coming out of the page, as shown, emerges from the centre of the square ABCD whose side is 3 m in length. ($a$) Using AMPÈRE'S LAW, find the average value along AB of the magnetic field component parallel to AB. ($b$) Find the magnitude and direction of the magnetic field at the midpoint of the line AB.




**ANSWER**: ($a$) The magnetic field does not point directly along the Ampèrian loop everywhere, nor is it constant in magnitude along the loop; but we are only interested in the average component parallel to AB. By symmetry, whatever $\vec{B}$ does along one side it does along the other three sides as well, so $\int_{\text{A}}^{\text{B}} \vec{B} \cdot d\vec{s} \equiv (3 \text{ m})\langle B_\| \rangle_{\text{AB}} = -\frac{1}{4}\oint_{\text{ABCD}} \vec{B} \cdot d\vec{s} = \frac{1}{4}\mu_0 I$, where the – sign indicates that the direction of $\vec{B}$ is generally from B to A (rather than from A to B) by the right-hand rule. All this gives $\langle B_\| \rangle_{\text{AB}} = \frac{1}{4}\mu_0 I/3 = -\frac{1}{4}4\pi \times 10^{-7} \times 2001/3$ or $\boxed{\langle B_\| \rangle_{\text{AB}} = -2.094 \times 10^{-4} \text{ T}}$. ($b$) The field at the midpoint of AB points toward A (by symmetry and the right-hand rule) and has a magnitude given by the usual formula for the field due to a long straight wire, namely $\boxed{B = \frac{\mu_0 I}{2\pi r} = \frac{2 \times 10^{-7} \times 2001}{3/2} = 2.67 \times 10^{-4} \text{ T}}$.