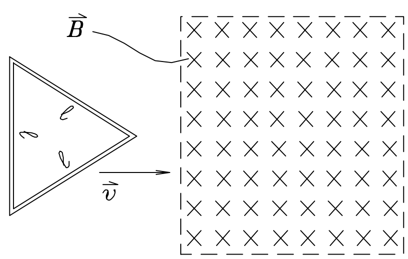
THE UNIVERSITY OF BRITISH COLUMBIA

Physics 108 Assignment # 8
SOLUTIONS:
FARADAY & INDUCTANCE

Wed. 2 Mar. 2005 — finish by Wed. 9 Mar.

1. **Earth's-Field Generator**: What is the maximum $\mathcal{EMF}$ that can be induced in a circular coil of 5000 turns and radius 50 cm by rotating it 30 times per second in the Earth's magnetic field in Vancouver ($B = 5 \times 10^{-5}$ T)?
ANSWER: The maximum $\mathcal{EMF}$ is produced when the axis of the coil rotates in a plane that includes the direction of the field — that is, at one point in the rotation the axis of the coil is parallel to $\vec{B}$ and $\frac{1}{4}$ of a period before or after that point it is perpendicular. The magnetic flux linking the coil is $\Phi = AB\cos\omega t$ where $A = \pi r^2 = 0.785$ m^2 and $\theta = \omega t$ is the angle between the coil axis and $\vec{B}$. The induced $\mathcal{EMF}$ is given by FARADAY'S LAW: $\mathcal{E} = -N d\Phi/dt = N\omega AB\sin\omega t$ where $N = 5000$ and $\omega = 2\pi \times 30 = 188.5$ s^{-1}, giving a maximum of $\mathcal{E}_{\max} = 5000 \times 188.5 \times 0.785 \times 5 \times 10^{-5}$ or $\boxed{\mathcal{E}_{\max} = 37.01 \text{ V}}$ when $\omega t = \pi/2$.

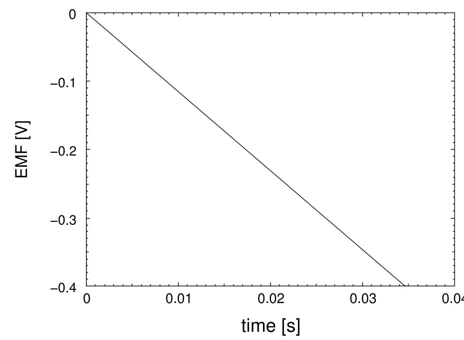
2. **Triangular Loop**: A wire loop in the shape of an equilateral triangle (length of a side $\ell = 0.2$ m) travelling at a constant speed $v = 5$ m/s moves, "pointy" end first, into a region where a uniform magnetic field $B = 0.4$ T points into the paper, as shown.



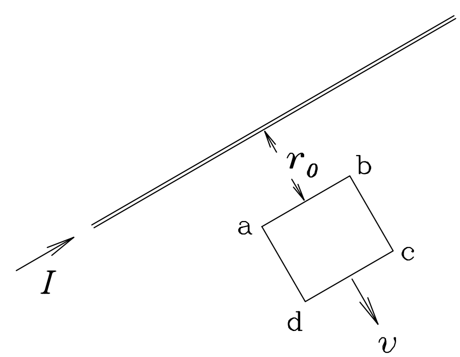
(*a*) Does current flow clockwise or counterclockwise (or not at all) around the triangular loop as it enters the field? **ANSWER**: The flux through the loop starts to increase; thus by LENZ'S LAW the current in the loop will flow so as to produce a field of its own in the opposite direction from the entering field. By the right-hand rule, the current must flow $\boxed{\text{counterclockwise}}$ to make a field *up* through the loop to counteract the incoming *down* field.

(*b*) What is the maximum induced $\mathcal{EMF}$ around the loop as it enters the field? **ANSWER**: The maximum $\mathcal{EMF}$ occurs when the last (and largest) slice of area enters the field — *i.e.* as the leftmost side is just entering. At that moment $d\Phi/dt = B dA/dt = B\ell v$ giving $\mathcal{E}_{\max} = -B\ell v = -0.4 \times 0.2 \times 5$ or $\boxed{\mathcal{E}_{\max} = -0.40 \text{ V}}$.

(*c*) Sketch the induced $\mathcal{EMF}$ around the loop as a function of time, from the time it begins to enter the field until it is entirely in the field. **ANSWER**: We could go through an algebraic derivation from geometrical principles, but it suffices to note that the rate of change of area in the field starts from zero as the tip enters the field and grows linearly with time until it reaches the maximum value as the last slice enters; after that the whole loop is in the field so the flux through the loop is constant and there is no induced $\mathcal{EMF}$. The length of the loop along the direction of motion is $\ell\cos 30° = 0.1732$ m so the entry takes $(\ell\cos 30°)/v = 0.03464$ s. After that the $\mathcal{EMF}$ abruptly returns to zero. See sketch.



3. **Moving Loop in Non-Uniform Field**: A long, straight, stationary wire carries a constant current of 150 A. Nearby **abcd**, a square loop 12 cm on a side, is moving *away* from the stationary wire (in a direction perpendicular to the wire) at a speed of $v = 6$ m/s. The long wire and the sides of the loop are all in a common plane; the near (**ab**) and far (**cd**) sides of the loop are parallel to the long wire and the other two sides (**bc** and **da**) are perpendicular to it. The near side (**ab**) is initially $r_0 = 15$ cm away from the long wire. Calculate the $\mathcal{EMF}$ around the square loop at this instant, assuming that the resistance of the loop is large enough that any actual current flowing around it produces a negligible magnetic flux. Also indicate the *direction* of the small current in side **cd**.



ANSWER: The field due to the wire is everywhere normal (perpendicular) to the plane of the loop and has a magnitude $B(r) = \mu_0 I/2\pi r$ as a function of the distance r away from the wire. The direction of $\vec{B}$ is "down" through the loop. Since B drops off as $1/r$, the magnitude of the flux through the loop will decrease as we pull it away from the wire. By LENZ'S LAW, the induced $\mathcal{EMF}$ around the loop will cause a current to flow that produces its own field in a direction that counteracts the change in flux — in this case, to reestablish the decreasing flux "down" through the loop; that means the current will flow around the loop clockwise (as viewed) and that $\boxed{\text{the current flows from } \mathbf{c} \text{ to } \mathbf{d}}$. (This answers the second part of the question.) Now we address the question of how much $\mathcal{EMF}$ is produced, using FARADAY'S LAW: For a given value of r_0, the flux through the loop is

$$\Phi = \frac{\mu_0}{2\pi} I\ell \int_{r_0}^{r_0+\ell} \frac{dr}{r} = \frac{\mu_0}{2\pi} I\ell \ln\left[\frac{r_0+\ell}{r_0}\right],$$

where $\ell = 0.12$ m is the length of a side.

Since $-\frac{d\Phi}{dt} = -\frac{d\Phi}{dr_0} \times \frac{dr_0}{dt} = -\frac{d\Phi}{dr_0} \times v$, FARADAY'S LAW gives

$$\mathcal{E} = -\frac{\mu_0}{2\pi} I \frac{\ell r_0}{r_0+\ell}\left[\frac{1}{r_0} - \frac{(r_0+\ell)}{r_0}\right] v = -\frac{\mu_0}{2\pi} I \frac{\ell r_0}{r_0+\ell}\left[\frac{1}{r_0+\ell} - \frac{1}{r_0}\right] v$$

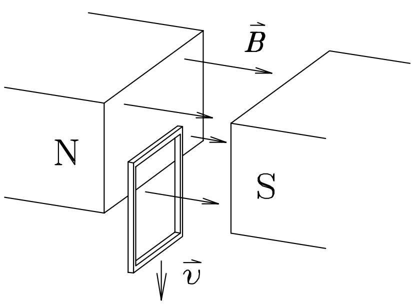
which, for $I = 150$, $r_0 = 0.15$, $\ell = 0.12$ and $v = 6$, gives $\mathcal{E} = -2 \times 10^{-7} \times 150 \times 0.15 \times 6[1/(0.15+0.12) - 1/0.15]$ or

$$\boxed{\mathcal{E} = 6.40 \times 10^{-5}\ \text{V}}.$$

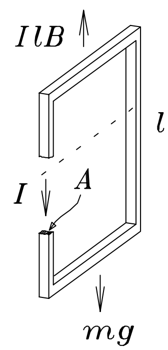
NOTE: There is a somewhat simpler way to do the same calculation, if we just go back to the idea that the Lorentz forces on the moving charges are equivalent to an electric field $\vec{\boldsymbol{E}}_{\text{eff}} = \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$ which makes a potential $\mathcal{E} = \int \vec{\boldsymbol{E}}_{\text{eff}} \cdot d\vec{\boldsymbol{s}}$ in wire segments **ab** and **cd**. This gives $\mathcal{E} = \ell v (B_{\mathbf{ab}} - B_{\mathbf{cd}})$ (where ℓ is the length of a side) which reduces to the same result without integrating and differentiating.

Challenge Problem for Extra Credit [20 marks]:

Dropping Frame: A square metallic frame is located, as shown, between the poles of an electromagnet, with its face perpendicular to $\vec{\boldsymbol{B}}$. The upper side is in a region of effectively uniform field with magnitude $B = 1.5$ T, while the lower side is outside the gap, where the field is essentially zero. If the frame is released and falls under its own weight, *determine the downward terminal velocity*. Assume the frame is made of aluminum (density 2.7 g/cm^3 and resistivity 2.8×10^{-6} Ω-cm). This problem requires careful thought. It is interesting that the terminal speed can be found with so little information about the metallic frame.



ANSWER: The total mass of the frame is $m = 4\ell A \rho_m$, where ℓ is the length of one side, A is the cross-sectional area of a side member (shown in a cutaway view in the sketch) and $\rho_m = 2.7 \times 10^3$ kg/m^3 is the mass density of Al. (We assume $\ell \gg \sqrt{A}$.) By the same token, the total electrical resistance around the frame is $R = 4\ell\rho_R/A$ where $\rho_R = 2.8 \times 10^{-8}$ Ω-m is the resistivity of Al. The area above the dashed line is linked with the magnetic field B and the region below is not, so as the frame drops the flux decreases, inducing an $\mathcal{EMF}$ $\mathcal{E} = -B\ell v$ that produces a current I that "tries" to replace the lost flux, as shown. The top member of the frame has I flowing perpendicular to the magnetic field and so experiences an upward force $F_M = I\ell B$ which will increase as I increases (which increases as $\mathcal{E}$ increases, which in turn increases as v increases) until it just balances the weight mg, after which acceleration ceases and we have the terminal velocity v_f given by $I\ell B = mg$. Put in $I = \mathcal{E}/R = B\ell v/R$ (dropping the $-$ sign) to get $B^2\ell^2 v/R = mg$ for



$v = v_f$, or $v_f = \dfrac{mgR}{\ell^2 B^2} = \dfrac{(4\ell A\rho_m)g(4\ell\rho_R/A)}{\ell^2 B^2} = \dfrac{16 g \rho_m \rho_R}{B^2}$ which in this case gives $v_f = \dfrac{16 \times 9.81 \times 2.7 \times 10^3 \times 2.8 \times 10^{-8}}{(1.5)^2}$ or

$\boxed{v_f = 0.00527\ \text{m/s}}$ (independent of ℓ or A).