THE UNIVERSITY OF BRITISH COLUMBIA

Physics 122 Assignment # 1 SOLUTIONS:
THERMAL PHYSICS

Fri. 04 Jan. 2002 — finish by Wed. 09 Jan.

1. "STAT - EC": Consider the following simplified model of a sort of *stock market*: A given stock $\mathcal{S}$ has a total of N shares on the market for a fixed price ε. At a given time, n of these shares are bought and the remaining $N-n$ are unwanted. Thus the net investment in $\mathcal{S}$ is $U = n\varepsilon$. [Here ε and U are measured in *monetary* units, say dollars; I have used the same notation as for *energy* for reasons that will soon become evident.] To keep things simple, we shall assume that the *price* ε of a given stock *does not change*. Further, let's make the outrageous assumption that the stock market as a whole is *a priori equally likely to be found in any one of the fully specified states accessible to it* — *i.e.* that a given amount of capital is equally likely to be distributed amongst the various stocks in any of the possible ways that give the same total.[1]

(*a*) Invent a general definition for an economic analogue of *temperature* τ [measured in monetary units] that has the desired predictive power: that (given our starting assumptions) capital will tend to flow spontaneously from stocks with *higher* τ into others with *lower* τ and will stop flowing between two stocks only when they are in "economic equilibrium" — *i.e.* when they have the same "economic temperature" τ.

ANSWER: This can be copied right off the handout; I just wanted you to give some thought to the implications. If $\Omega(U) \equiv \Omega(n, N)$ is the number of different ways the total capital investment $U = n\varepsilon$ in stock $\mathcal{S}$ can be redistributed among the N shares available ($N > n$) of that stock, then the ENTROPY of stock $\mathcal{S}$ is $\sigma = \ln \Omega$. We even have a formula for $\Omega(n, N)$ given the above specifications, namely the *binomial distribution*, but this part of the question is completely general and you need not work out the actual specific result, just define the inverse economic temperature as $\boxed{\frac{1}{\tau} \equiv \frac{\partial \sigma}{\partial U},}$ in terms of which the above-mentioned predictive power comes automatically!

Just for fun, I will go ahead and derive the explicit U-dependence of τ for the binomial distribution: $\Omega(n, N) = \dfrac{N!}{n!(N-n)!}$ or $\Omega(U) = \dfrac{(2\hat{U}/\varepsilon)!}{(U/\varepsilon)![(2\hat{U} - U)/\varepsilon]!}$ where $\hat{U} \equiv \frac{1}{2}N\varepsilon$ is the energy giving the largest possible Ω. For large N this can be approximated by a gaussian distribution: $\Omega(U) \approx \Omega(\hat{U}) \exp\left[-2(U-\hat{U})^2/N\varepsilon^2\right]$, giving $\sigma(U) \equiv \ln \Omega(U) = c - (U - \hat{U})^2/\hat{U}\varepsilon$. Thus $\dfrac{1}{\tau} = \dfrac{\partial \sigma}{\partial U} = -2\dfrac{(U-\hat{U})}{\hat{U}\varepsilon} = 2\dfrac{1 - U/\hat{U}}{\varepsilon}$ or $\tau = \dfrac{\varepsilon}{2(1 - U/\hat{U})}$. Note that τ has the same dimensions as ε, as expected. For your own amusement, consider how the economic temperature of stock $\mathcal{S}$ behaves when half or more of its available shares are sold!

(*b*) Now assume that the entire market is in "economic equilibrium" and is so much larger than any of its parts that we may treat it as a "capital reservoir" $\mathcal{R}$ at an "economic temperature" of $\tau =$\$100. Consider one share of one stock, valued at $\varepsilon_1 =$\$200: What is the probability that it will be bought at any given time?

ANSWER: Again, this is right out of the handout: we have now the analogue of the CANONICAL ENSEMBLE, in which the probability of one fully specified state of a "microsystem" (in this case a single share of a given stock) follows the BOLTZMANN DISTRIBUTION: $\mathcal{P}_\alpha \propto e^{-\varepsilon_\alpha/\tau}$. In order to convert the $\propto$ symbol into an $=$ sign, we need to *normalize* the probability distribution: all the possibilities together must add up to a probability of unity. In this case there are only two possibilities: either the stock is *bought* (investment ε, probability $\mathcal{P}_b = Ce^{-\varepsilon/\tau}$) or it is *not bought* (investment 0, probability $\mathcal{P}_{nb} = Ce^{-0/\tau} = C$), where the constant of proportionality C must be adjusted to make the total come out to $1 = \mathcal{P}_b + \mathcal{P}_{nb} = C(1 + e^{-\varepsilon/\tau})$. Thus $C = 1/(1+e^{-\varepsilon/\tau})$ and we have $\mathcal{P}_b = \dfrac{e^{-\varepsilon/\tau}}{1 + e^{-\varepsilon/\tau}} = \dfrac{1}{e^{+\varepsilon/\tau} + 1}$, in this case $\boxed{\mathcal{P}_b = \dfrac{1}{e^{\$200/\$100} + 1} = \dfrac{1}{e^2 + 1} = 0.1192.}$

(*c*) Assuming that $\mathcal{R}$ is also huge compared to the entire offering of $N_1 = 1000$ shares of stock $\mathcal{S}_1$ valued at $\varepsilon_1 =$\$200, what is the expected total investment U_1 in $\mathcal{S}_1$ when $\tau =$\$100?

[1] This is not consistent with current economic theory, which focusses on "rational agents." Here we assume totally mindless, random investment decisions.

ANSWER: If the probability of *any one* share of $\mathcal{S}$ being bought is $\mathcal{P}_b = 1/(e^2+1)$, then the expected average total investment U in $\mathcal{S}$ is $\langle U \rangle = N\mathcal{P}_b\varepsilon = 1000 \times \dfrac{1}{e^2+1} \times \$200$ or $\boxed{\langle U \rangle = \$23,841.}$

(*d*) If the economic temperature drops to $\tau =\$50$, which stock will be likely to have the most capital U invested in it, $\mathcal{S}_1$ with $N_1 = 1000$ shares at $\varepsilon_1 =\$200$ per share or $\mathcal{S}_2$ with $N_2 = 1000$ shares at $\varepsilon_2 =\$100$ per share?

ANSWER: As in the previous part, $\langle U_1 \rangle = N_1 \times \dfrac{1}{e^{200/50}+1} \times \varepsilon_1 = 1000 \times \dfrac{1}{e^4+1} \times \$200 = \$3,597$ whereas $\langle U_2 \rangle = N_2 \times \dfrac{1}{e^{100/50}+1} \times \varepsilon_2 = 1000 \times \dfrac{1}{e^2+1} \times \$100 = \$11,920$. Thus the stock selling for the lower price will tend to attract the larger *net investment* for the same number of shares, even though each share is sold for less, as long as both are *expensive* — *i.e.* selling for well above the market's economic temperature τ. (If $\varepsilon < \tau$ then almost all shares of the stock will be sold and the stock with the larger price per share will attract a larger net investment. Notice how important it is to be aware of the market's "temperature.") Of course, all this is founded on several assumptions — totally mindless investment decisions and the consequent lack of "feedback" affecting the prices of stocks — which cannot possibly describe the behaviour of a real stock market ... can they?

2. ORTHO- *vs.* PARA-HYDROGEN: Molecular hydrogen, H_2, consisting of two protons bound together with two electrons, can form in either the "singlet" state called *parahydrogen*, in which the total spin (intrinsic angular momentum) of the molecule is zero, or in any one of three "triplet" states of *orthohydrogen*, in which the proton spins combine to make a total spin of $1\hbar$ (the fundamental unit of angular momentum). For this problem, all you need to know is that the three triplet states are *degenerate* — *i.e.* they all have the same *energy* relative to the singlet state, namely $\varepsilon_3 = 2.375 \times 10^{-21}$ J. (The energy ε_1 of the singlet state can be taken to be zero, for reference.) Assume that the spin degrees of freedom of the H_2 molecules are unaffected by, but are in thermal equilibrium with, all their other degrees of freedom (like translational, rotational or vibrational). In this case, what *fraction* f_3 of H_2 molecules will be found (on average) in *ortho* states

(*a*) at room temperature (300 K)?

ANSWER: As in the previous example, the probability of a state $|\alpha\rangle$ of energy ε_α being occupied in thermal equilibrium at temperature τ is given by the Boltzmann distribution, $\mathcal{P}_\alpha = Ce^{-\varepsilon_\alpha/\tau}$, where the constant of proportionality C is to be determined by *normalization*.[2] There are three states of energy ε_3 and one of zero energy, so for normalization we must have $1 = C + 3Ce^{-\varepsilon_3/\tau}$ or $C = \dfrac{1}{1+3e^{-\varepsilon_3/\tau}}$. The *fraction* of H_2 molecules in all three triplet states combined is the sum of the three (equal) probabilities of any given molecule being in any of these three degenerate states: $f_3 = 3Ce^{-\varepsilon_3/\tau} = \dfrac{3e^{-\varepsilon_3/\tau}}{1+3e^{-\varepsilon_3/\tau}} = \dfrac{3}{e^{\varepsilon_3/\tau}+3}$. For $\tau = 300k_B = 4.142 \times 10^{-21}$ *J*, $\varepsilon_3/\tau = 0.5734$ and $e^{\varepsilon_3/\tau} = 1.774$, giving $\boxed{f_3 = \dfrac{3}{1.774+3} = 0.628}$

(*b*) at the boiling point of liquid nitrogen at atmospheric pressure (77 K)?

ANSWER: The same formula applies here, with $\tau = 77k_B = 1.063 \times 10^{-21}$ *J*, $\varepsilon_3/\tau = 2.234$ and $e^{\varepsilon_3/\tau} = 9.337$, giving $\boxed{f_3 = \dfrac{3}{9.337+3} = 0.243}$

(*c*) at the freezing point of molecular hydrogen at atmospheric pressure (14 K)?

ANSWER: ...and again here, with $\tau = 14k_B = 1.933 \times 10^{-22}$ *J*, $\varepsilon_3/\tau = 12.29$ and $e^{\varepsilon_3/\tau} = 2.168 \times 10^5$, giving $\boxed{f_3 = \dfrac{3}{2.168 \times 10^5 + 3} = 1.384 \times 10^{-5}}$

[2] In this case, of course, ε is an actual *energy* and τ is a normal *temperature* measured in regular energy units. (Back to the "real world" of thermal physics.)