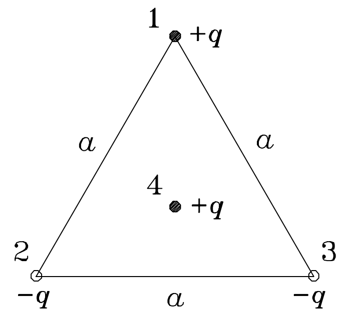
THE UNIVERSITY OF BRITISH COLUMBIA

Physics 122 Assignment # 3 SOLUTIONS:
ELECTRIC CHARGE

Wed. 16 Jan. 2002 — finish by Fri. 25 Jan.



1. TRIANGLE of CHARGES: Derive an expression for the *work* required to bring four charges of equal magnitude but different signs (as shown) together into an equilateral triangle of side a with one charge at the centre of the triangle. (Initially the charges are all infinitely far apart.)

ANSWER: The work required to bring the charges together from infinity is the same as the net potential energy U_E, which is turn is the sum of the potentials for each *pair* of particles:

$$U_E = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1q_2}{r_{12}} + \frac{q_2q_3}{r_{23}} + \frac{q_1q_3}{r_{13}} + \frac{q_1q_4}{r_{14}} + \frac{q_2q_4}{r_{24}} + \frac{q_3q_4}{r_{34}}\right],$$

where the labels "1,2,3,4" are shown on the Figure. (Note that no pair is counted twice.) Now, $q_1 = q_4 = +q$ and $q_2 = q_3 = -q$; meanwhile $r_{12} = r_{23} = r_{13} = a$ and $r_{14} = r_{24} = r_{34} = a/\sqrt{3}$. So

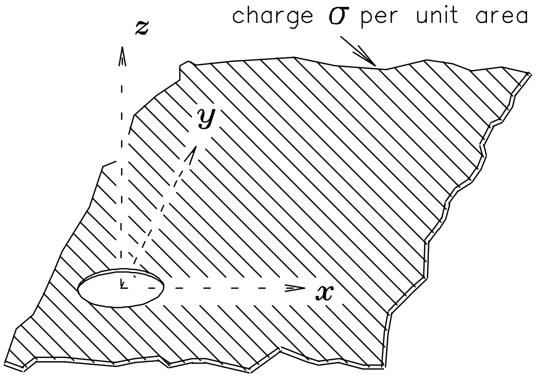
$U_E = \frac{q^2}{4\pi\epsilon_0 a}\left[-1+1-1+\sqrt{3}-\sqrt{3}-\sqrt{3}\right]$ or $\boxed{U_E = -\frac{q^2}{4\pi\epsilon_0 a}\left(1+\sqrt{3}\right).}$ Numerically,

$\boxed{U_E\ [J] = -2.732 \times 10^{10}\,(q\ [J])^2/(a\ [m]).}$ (Either form is acceptable.)

2. LINES of CHARGE: An infinite line of charge with linear charge density $\lambda = 5 \times 10^{-10}$ C/m lies along the x-axis ($y = 0$, $z = 0$ and $-\infty < x < +\infty$). A second infinite line of charge with exactly the opposite charge density lies along the z axis. What are the x, y and z components of the resultant *electric field* at the point $(x, y, z) = (4, 4, 4)$ m?

ANSWER: The text gives the result $\vec{E} = \hat{r}\lambda/2\pi\epsilon_0 r$ for the electric field due to a line of charge, where the $\vec{r}$ is a vector from the line to the point at which $\vec{E}$ is to be evaluated, perpendicular to the line. (By symmetry, there is no other possible direction for $\vec{E}$ to point!) Thus the field $\vec{E}_1$ due to the first line of charge points directly away from the x axis [at the specified point this will be in a direction $\hat{r}_1 = (\hat{\jmath} + \hat{k})/\sqrt{2}$] and has a magnitude $E_1 = \lambda/2\pi\epsilon_0 r_1$ with $r_1 = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ m, so $E_1 = \frac{2\times5\times10^{-10}}{2\pi\epsilon_0\times4\sqrt{2}} = 1.589$ N/C. The second line of charge is the same distance away from our test point ($r_2 = r_1$) and the magnitude of the charge per unit length is the same (λ) so the magnitude of the second contribution to the electric field is the same as that of the first ($E_2 = E_1$) but in this case the charge is opposite (negative) so $\vec{E}_2$ points *toward* the other line: $\vec{E}_2 = -\hat{r}_2 E_2$ where $\hat{r}_2 = (\hat{\imath} + \hat{\jmath})/\sqrt{2}$. Thus in total $\vec{E} = \vec{E}_1 + \vec{E}_2 = [\hat{\jmath} + \hat{k} - \hat{\imath} - \hat{\jmath}] \times 1.589/\sqrt{2}$ or $\boxed{E_x = -1.123 \text{ N/C},\ E_y = 0 \text{ and } E_z = 1.123 \text{ N/C}).}$

3. PLANE of CHARGE: A large flat non-conducting surface carries a uniform charge density $\sigma = 4.0 \times 10^{-9}$ C/m^2. A small circular hole has been cut out of the middle of this sheet of charge, as shown on the diagram. Ignoring "fringing" of the field lines around all distant "outside" edges, calculate the *electric field* at point P a distance $z = 1.0$ m up from the centre of the hole along its axis. The radius of the hole is $R = 0.4$ m.



small sample area
from infinite plane.

ANSWER: We can treat this as the superposition of a uniformly charged sheet (without a hole), which gives a uniform electric field $E_1 = \dfrac{\sigma}{2\epsilon_0}$ normal to the surface, and a negatively charged disc at the position of the hole, which produces a field $E_2 = -\dfrac{\sigma}{2\epsilon_0}\left[1 - \dfrac{z}{\sqrt{z^2+R^2}}\right]$ (parallel to the axis) at a point z away on the axis. Thus
$E = E_1 + E_2 = \dfrac{\sigma}{2\epsilon_0}\left(1 - \left[1 - \dfrac{z}{\sqrt{z^2+R^2}}\right]\right) = \dfrac{\sigma}{2\epsilon_0}\left(\dfrac{z}{\sqrt{z^2+R^2}}\right) = \dfrac{4.0 \times 10^{-9}}{2 \times 0.8854 \times 10^{-11}} \times \dfrac{1.0}{\sqrt{1.0^2 + 0.4^2}}$ or
$\boxed{E = 209.7\ V/m}$ directed along the positive z axis.

4. EXTRA ELECTRONS: The Earth can be regarded as a spherical conductor from the point of view of free electrons. *How many excess electrons* could the Earth hold? (Assume that only electrostatic and gravitational forces are involved.)

ANSWER: The N excess electrons repel each other and so "try" to get as far from one another as possible; they therefore accumulate uniformly over the surface of the Earth (radius R_E). By the SHELL THEOREM (which works just as well for the electrostatic force as for the gravitational force, since both obey inverse square laws) the resultant electric field will be the same as if all the charge were concentrated at the Earth's centre. Thus the *electrical* force on any *one* electron will be $F_E = eE = \dfrac{1}{4\pi\epsilon_0}\dfrac{Ne^2}{R_E^2}$ (away from the Earth) whereas the *gravitational* force on that same electron will be $F_G = m_e g = G\dfrac{M_E m_e}{R_e^2}$ (toward the Earth). When we have as many "extra" electrons as possible, the two forces will just cancel ($F_E = F_G$), giving $\dfrac{1}{4\pi\epsilon_0}Ne^2 = GM_E m_e$ or
$N = \dfrac{4\pi\epsilon_0 G M_E m_e}{e^2} = \dfrac{4\pi\epsilon_0 g m_e R_E^2}{e^2} = \dfrac{9.11 \times 10^{-31} \times (6.37 \times 10^6)^2}{8.99 \times 10^9 \times (1.602 \times 10^{-19})^2}$ or $\boxed{N = 1.574 \times 10^{12}.}$ (Note that this is only 2.52×10^{-7} C! Since it is possible to pick up as well as lose electrons from the solar wind, the Earth remains electrically neutral to uncanny precision!)