THE UNIVERSITY OF BRITISH COLUMBIA

Physics 122 Assignment # 4
SOLUTIONS:
GAUSS' LAW

Fri. 25 Jan. 2002 — finish by Fri. 1 Feb.

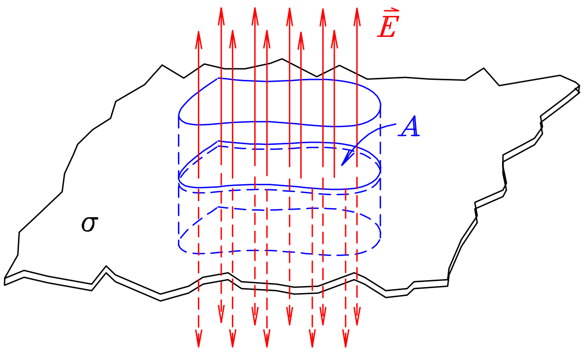
1. GAUSS' LAW FOR A SHEET OF CHARGE: Imagine an infinite plane sheet of electric charge with $dQ/dA = \sigma$ units of charge per unit surface area. *Using your own words and drawings,*[1]

(a) Using only (i) the SUPERPOSITION PRINCIPLE for the total electric field due to an assembly of electric charges, plus (ii) simple SYMMETRY arguments, deduce the *direction* of the electric field due to the plane of charge.

ANSWER: Pick a point P a perpendicular distance x away from the sheet. Now pick a small area element dA anwhere on the sheet and draw a vector $\vec{r}$ from dA to P. The electric field at P due to dA will have a magnitude dE and a direction along $\vec{r}$. By symmetry, there is another area element dA' of the same size (enclosing the same charge dQ) in the sheet on the other side of P whose *parallel* component exactly cancels that due to dA. Since the whole surface can be composed from such pairs, the only direction the electric field could have is *normal to the sheet*. You could also appeal to the infiniteness of the sheet: if there were to be a "preferred direction" parallel to the sheet, *how would you pick it* with nothing to refer to? If the charge of the sheet is positive, the field will point away from it; if negative, towards it. By the same argument, the field must be the same at any other point the same distance x away from the sheet.

(b) Using the preceding result plus the general ideas of GAUSS' LAW and simple geometry, deduce the dependence of the *magnitude* E of the electric field upon x, the *perpendicular distance* away from the plane.

ANSWER: Draw an arbitrary perimeter around some area A on the sheet and duplicate it a distance x on either side of the sheet. These flat "endcaps" define an imaginary "pillbox" whose sides are normal to the plane (and therefore intercept no electric field lines) and whose endcaps of area A each intercept the same electric flux EA where E is the field strength at distance x. This pillbox encloses a net charge $A\sigma$ and so Gauss' law gives $2\epsilon_0 EA = A\sigma$ or (cancelling out the A) $\boxed{E = \dfrac{\sigma}{2\epsilon_0}}$. Note that E is *independent* of x. The field is *uniform* (constant).



(c) Show that your result agrees with the x-dependence of the electric field on axis due to a uniform *disc* of charge when $x \ll R$, the radius of the disc.

ANSWER: $E(x) = \dfrac{\sigma}{2\epsilon_0}\left[1 - \dfrac{x}{\sqrt{x^2+R^2}}\right]$ for a disc. For $R \gg x$ the second term becomes negligible and we recover the result for an infinite plane.

2. ATOMS AS SPHERES OF CHARGE: In Rutherford's work on α particle scattering from atomic nuclei, he regarded the atom as having a pointlike positive charge of $+Ze$ at its centre, surrounded by a spherical volume of radius R filled with a uniform charge density that makes up a total charge $-Ze$. In this simple model, calculate the electric field strength E and the electric potential ϕ as functions of radius r and various constants. *Plot* your results for $0 < r \leq 2R$. (Choose $\phi \underset{r\to\infty}{\longrightarrow} 0$.)

ANSWER: By Gauss' Law, $\Phi_E \equiv \oiint \vec{E}\cdot d\vec{A} = 4\pi k Q_{\rm enc} = Q_{\rm enc}/\epsilon_\circ$ where $Q_{\rm enc}$ is the total charge inside the closed Gaussian surface: if we use a sphere of radius $r < R$ then $Q_{\rm enc} = +Ze$ (from protons) $- Ze(r/R)^3$ (from electrons). By spherical symmetry, $\vec{E}$ points radially outward (normal to the surface) and is the same strength all over the sphere, so Gauss' Law yields $\Phi_E = (E)(4\pi r^2) = Ze\left[1 - (r^3/R^3)\right]/\epsilon_\circ$, giving

$$\boxed{E = \left(\frac{Ze}{4\pi\varepsilon_0}\right)\left(\frac{1}{R^2}\right)\left[\frac{R^2}{r^2} - \frac{r}{R}\right] \text{ for } r < R}.$$

At $r = R$ this goes to zero; for $r > R$ there are equal amounts of positive and negative charge enclosed, so Gauss' Law tells us that $\boxed{E(r > R) = 0}$.

[1] The full derivation can be found in any textbook; however, you will not really *own* Gauss' Law until you have expressed and used it *your own way*. This is true for almost everything, but Gauss' Law is so important (and so *simple!*) that you should not miss this chance to make it yours.

Now to calculate the potential ϕ:

If we take the potential to be zero for $r \to \infty$, $\boxed{\phi(r > R) = 0}$, then integrating $-E(r')dr'$ from $r' = R$ to $r' = r$ (the same thing as integrating $E(r')dr'$ from $r' = r$ to $r' = R$) gives

$$\phi(r < R) = \left(\frac{Ze}{4\pi\varepsilon_0}\right) \int_r^R \left(\frac{1}{r'^2} - \frac{r'}{R^3}\right) dr' \; .$$

The integral can be split into two parts,

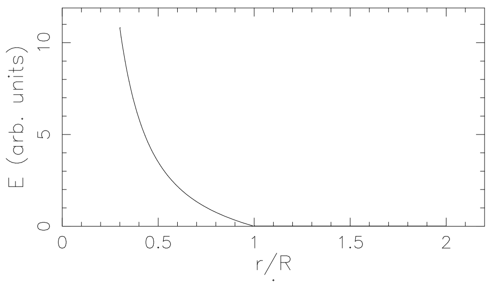
$$\int_r^R \frac{dr'}{r'^2} = \left[-\frac{1}{r'}\right]_r^R = \frac{1}{r} - \frac{1}{R} \qquad \text{and}$$

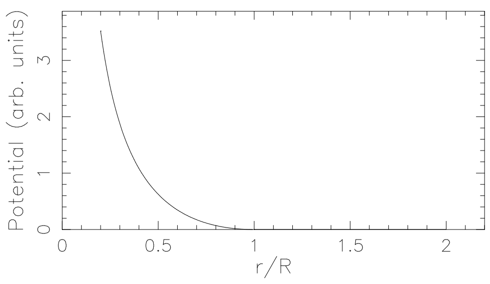
$$-\int_r^R \frac{r'\,dr'}{R^3} = -\left[\frac{1}{2}\frac{r'^2}{R^3}\right]_r^R = \frac{1}{2}\left\{\frac{r^2}{R^3} - \frac{R^2}{R^3}\right\}$$

Although these are not hard integrals, one can easily fall into confusion by not thinking carefully about the *limits* of integration. Since $R^2/R^3 = 1/R$, we can combine the two results to give

$$\boxed{\phi(r < R) = \left(\frac{Ze}{4\pi\varepsilon_0}\right)\left(\frac{1}{R}\right)\left[\frac{R}{r} + \frac{1}{2}\frac{r^2}{R^2} - \frac{3}{2}\right]} \; .$$

The plots looks like this:



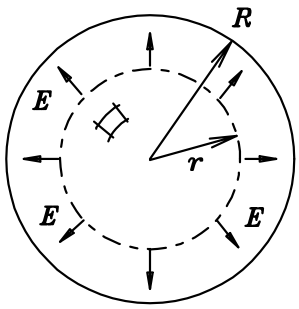


The two graphs have qualitatively similar behaviour — each "blows up" as $r \to 0$ and drops to zero at $r = R$ — but they really are different functions.

3. FIELD WITHIN A UNIFORM CHARGE DISTRIBUTION: The textbook shows how to use GAUSS' LAW to derive the radial (r) dependence of the electric field $E(r > R)$ *outside* charge distributions of spherical, cylindrical or planar symmetry, where R is the distance the charge distribution extends from the centre of symmetry — the radius of a charged *sphere* or *cylinder*, or half the thickness of an infinite *slab* of charge, respectively. Use similar arguments to show that, for each of these cases (a sphere, cylinder or a slab of uniform charge density), the electric field $E(r < R)$ *inside* the charge distribution is given in terms of the field $E(R)$ *at the boundary* of the charge distribution by

$$E(r < R) = \left(\frac{r}{R}\right) E(R) \; .$$

ANSWER:
Sphere:

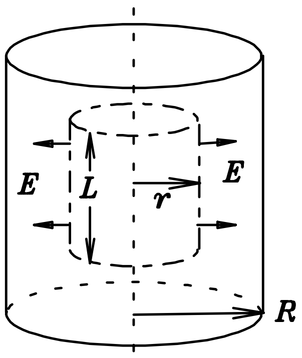


$$\text{charge density} \quad \rho = \frac{3Q}{4\pi R^3}$$

$$\epsilon_\circ(4\pi r^2 E) = Q_{\rm encl} = \frac{4}{3}\pi r^3 \rho = \left(\frac{r^3}{R^3}\right) Q$$

$$E(r) = \frac{Q}{4\pi\epsilon_\circ}\frac{r}{R^3} = E(R)\left(\frac{r}{R}\right) \; . \quad \mathcal{QED}$$

Cylinder:

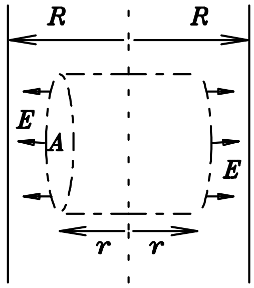


$$\lambda = \pi R^2 \rho \Longrightarrow \rho = \frac{\lambda}{\pi R^2}$$

$$\epsilon_\circ(2\pi r L E) = Q_{\rm encl} = \pi r^2 L \rho = L\lambda\left(\frac{r^2}{R^2}\right)$$

$$E(r) = \frac{\lambda}{2\pi\epsilon_\circ}\frac{r}{R^2} = E(R)\left(\frac{r}{R}\right) \; . \quad \mathcal{QED}$$

Slab:



$$\sigma = 2R\rho \Longrightarrow \rho = \frac{\sigma}{2R}$$

$$\epsilon_\circ(2AE) = Q_{\rm encl} = 2Ar\rho = A\sigma\left(\frac{r}{R}\right)$$

$$E(r) = \frac{\sigma}{2\epsilon_\circ}\frac{r}{R} = E(R)\left(\frac{r}{R}\right) \ . \quad \mathcal{QED}$$