THE UNIVERSITY OF BRITISH COLUMBIA

Physics 122 Assignment # 5 SOLUTIONS: POTENTIAL & CAPACITANCE

Fri. 1 Feb. 2002 — finish by Fri. 8 Feb.

1. CLASSICAL RADIUS OF THE ELECTRON: You are probably familiar with Einstein's famous equation $E = mc^2$. If m is the mass of an electron and E is the electrostatic potential energy required to "assemble" the electron from bits of charge infinitely distant from each other into a uniform spherical shell of radius r_0 and net charge e, find the numerical value of r_0 in meters.[1]

ANSWER: Start with no charge, then bring successive bits dQ in to add uniformly to the charge Q of the shell. A given bit of charge acquires an electrostatic potential energy $dE = k_E Q dQ/r_0$ in the process. Thus the total energy E required to assemble the shell is $E = k_E/r_0 \int_0^e Q dQ = \frac{1}{2} k_E e^2/r_0$. If we set this equal to mc^2 we get $r_0 = \frac{1}{2} k_E \frac{e^2}{mc^2} = \frac{(8.998 \times 10^9)(1.6022 \times 10^{-19})^2}{2(9.11 \times 10^{-31})(2.998 \times 10^8)^2}$ or $\boxed{r_0 = 1.409 \times 10^{-15} \text{ m}}$. The conventional value (actually the Compton wavelength of the electron) is twice as big, 2.818 fm, where "fm" stands for "femtometers" or "fermis" (named after Enrico Fermi); both are the same as 10^{-15} m.

2. CAPACITOR WITH INSERT: Suppose we have a capacitor made of two large flat parallel plates of the same area A (and the same shape), separated by an air gap of width d. Its capacitance is C. Now we slip another planar conductor of width $d/2$ (and the same area and shape) between the plates so that it is centred halfway in between. What is the capacitance C' of the new system of three conductors, in terms of the capacitance C of the original pair and the other parameters given? (Neglect "edge effects" and any dielectric effect of air.)

ANSWER: The original capacitance was $C = \epsilon_\circ A/d$. The capacitor with the insert as shown is equivalent to two identical capacitors in series, each of which has a gap of $d/4$ between its plates, so that $C_1 = C_2 = 4C$. The equivalent capacitance of two capacitors in series is given by $1/C' = 1/C_1 + 1/C_2 = 2/4C = 1/2C$. Thus $\boxed{C' = 2C}$.

3. CUBIC CAPACITOR Suppose we take a roll of very thin (50 μm) copper sheet and a roll of 150 μm thick strontium titanate dielectric (see Table 29-2 on p. 671 of the textbook) and form a capacitor as follows: cut the sheets into strips 5 cm wide and sandwich the dielectric sheet between two sheets of copper. Then fold the sandwich back and forth to fill a cube 5 cm on each side. Assuming that we can press the layers together so that there are no empty spaces, find:

(*a*) the capacitance of the resulting cube-shaped capacitor; **ANSWER**: Each layer consists of 2 sheets of Cu and 1 sheet of $SrTiO_3$ and so is 250 μm or 2.5×10^{-4} m thick. The number of layers is therefore $N = 5 \times 10^{-2}/2.5 \times 10^{-4} = 200$ and the total area is $A = N \times 0.05 \times 0.05 = 0.5$ m^2. The capacitance is $C = \kappa\epsilon_\circ A/d$ where $d = 150$ μm. The dielectric constant of $SrTiO_3$ is (from Table 27-2) $\kappa = 310$, so $C = 310 \times 8.85 \times 10^{-12} \times 0.5/1.5 \times 10^{-4}$ or $\boxed{C = 9.145 \times 10^{-6} \text{ F or } 9.145\ \mu\text{F}}$.

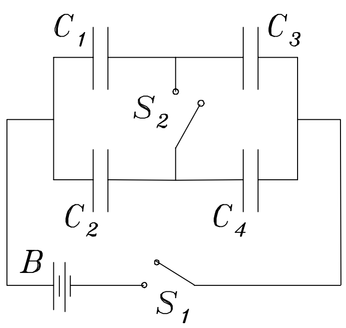
(*b*) the maximum charge it will hold without breaking down; **ANSWER**: Table 27-2 lists the dielectric strength of $SrTiO_3$ as 8 kV/mm, which is the same as 8×10^6 V/m. Our dielectric sheet has a thickness of 1.50×10^{-4} m, so the breakdown voltage is 1200 V. At that potential, $Q = CV$ gives $\boxed{Q = 0.01097 \text{ C}}$.[2]

(*c*) the total energy we can store in this small cube. **ANSWER**: The total energy stored in a capacitor is given by $\boxed{U = \frac{1}{2}CV^2 = 6.584 \text{ J}}$.

[1] The value you calculate will not agree with the value you look up; this is because the r_0 listed in textbooks is actually the Compton radius of the electron and has a completely different meaning. Nevertheless, numerous texts glibly describe r_0 as defined in this problem. The amazing thing is that the two values are so close!

[2] Note: the abbreviation C (for Coulombs) can often be distinguished from the symbol C (for capacitance) only by context. Here you can tell them apart by *font*, but in handwritten solutions this is impractical. In case of ambiguity one can always write out "Coulombs" without abbreviation.

4. ARRAY of CAPACITORS: The battery B supplies 6 V. The capacitances are $C_1 = 2.0\ \mu$F, $C_2 = 1.0\ \mu$F, $C_3 = 4.0\ \mu$F and $C_4 = 3.0\ \mu$F. *(a)* Find the charge on each capacitor when switch S_1 is closed but switch S_2 is still open. ANSWER: Let Q_i denote the charge on the i^{th} capacitor C_i. From charge conservation we have $Q_1 = Q_3$ and $Q_2 = Q_4$. Both pairs of capacitors in series (1 and 3; 2 and 4) must make up the full voltage: $V_B = Q_1/C_1 + Q_3/C_3 = Q_2/C_2 + Q_4/C_4$. Therefore $V_B = Q_1[1/C_1 + 1/C_3] = Q_2[1/C_2 + 1/C_4]$ yielding $Q_1 = Q_3 = 6/(10^6/2.0 + 10^6/4.0)$ and $Q_2 = Q_4 = 6/(10^6/1.0 + 10^6/3.0)$ or $\boxed{Q_1 = Q_3 = 8.0 \times 10^{-6}\ \text{C}}$ and $\boxed{Q_2 = Q_4 = 4.5 \times 10^{-6}\ \text{C}}$.



(b) What is the charge on each capacitor if S_2 is also closed? **ANSWER**: Now C_1 and C_2 are effectively just one big capacitor $C_{12} = C_1 + C_2 = 3.0\ \mu$F and similarly for $C_{34} = C_3 + C_4 = 7.0\ \mu$F. Charge conservation now requires $Q_{12} \equiv Q_1 + Q_2 = Q_{34} \equiv Q_3 + Q_4$ and the two effective capacitors in series must make up the full voltage: $V_B = Q_{12}/C_{12} + Q_{34}/C_{34}$. Thus $V_B = Q_{12}[1/C_{12} + 1/C_{34}]$ giving $Q_{12} = Q_{34} = 6/(10^6/3.0 + 10^6/7.0) = 12.6 \times 10^{-6}$ C. Meanwhile the voltage across C_1 must be the same as that across C_2: $Q_1/C_1 = Q_2/C_2 \Longrightarrow Q_2 = Q_1(C_2/C_1) = \frac{1}{2}Q_1 \Longrightarrow Q_{12} = \frac{3}{2}Q_1$ or $\boxed{Q_1 = \frac{2}{3}Q_{12} = 8.4 \times 10^{-6}\ \text{C}}$ and $\boxed{Q_2 = Q_{12} - Q_1 = 4.2 \times 10^{-6}\ \text{C}}$. Similarly, $Q_3/C_3 = Q_4/C_4 \Longrightarrow Q_4 = Q_3(C_4/C_3) = \frac{3}{4}Q_3 \Longrightarrow Q_{34} = \frac{7}{4}Q_3$ or $\boxed{Q_3 = \frac{4}{7}Q_{34} = 7.2 \times 10^{-6}\ \text{C}}$ and $\boxed{Q_4 = Q_{34} - Q_3 = 5.4 \times 10^{-6}\ \text{C}}$.

5. THUNDERCLOUD CAPACITOR: A large thundercloud hovers over the city of Vancouver at a height of 2.0 km. Between the cloud and the ground (both of which we may treat as parallel conducting plates, neglecting edge effects) the electric field is about 200 V/m. The cloud has a horizontal area of 200 km^2.

(a) Estimate the number of Coulombs [C] of positive charge in the cloud, assuming that the ground has the same surface density of negative charge. **ANSWER**: The electric field between two flat plates with surface charge densities $\pm\sigma$ is given by $E = \sigma/\epsilon_\circ$. Thus $\sigma = \epsilon_\circ E = 8.85 \times 10^{-12} \times 200 = 1.77 \times 10^{-9}$ C/m^2. Over an area of $A = 200 \times 10^6 = 2 \times 10^8$ m^2, this gives a total charge of $\boxed{Q = \sigma A = 0.354\ \text{C}}$.

(b) Estimate the number of joules [J] of energy contained in the air between the cloud and the ground. **ANSWER**: The energy density stored in an electric field is given by $U/V = \frac{1}{2}\epsilon_\circ E^2 = 0.5 \times 8.85 \times 10^{-12} \times (200)^2 = 0.177 \times 10^{-6}$ J/m^3. The volume between the cloud and the ground is $V = 2000 \times 2 \times 10^8 = 4 \times 10^{11}$ m^3, so $\boxed{U = 0.708 \times 10^5\ \text{J}}$.