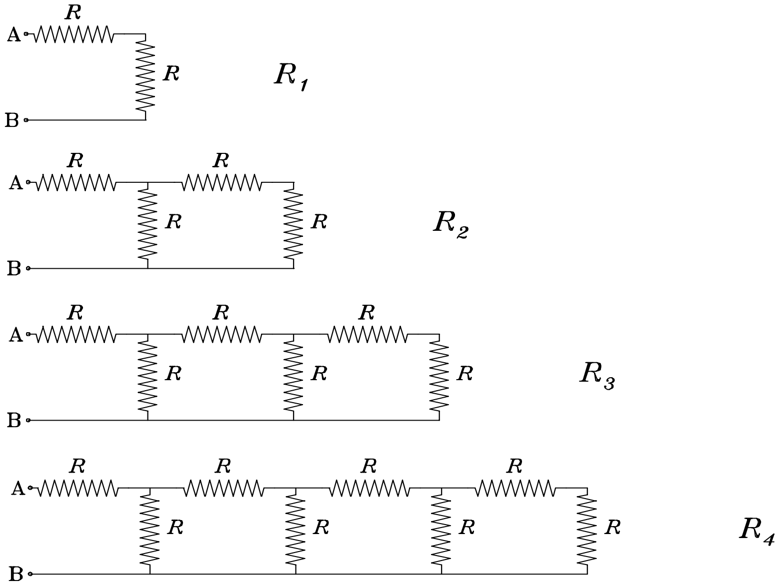
The University of British Columbia

Physics 122 Assignment # 6 SOLUTIONS:
CURRENT, RESISTANCE & DC CIRCUITS

Fri. 8 Feb. 2002 — finish by Fri. 15 Feb.

1. **TRIUMF POWER USE:** The electromagnet that generates the magnetic field for the world's largest cyclotron at TRIUMF has conductors made of aluminum (resistivity $\rho = 2.8 \times 10^{-8}\ \Omega\mathrm{m}$) wound in a circle of radius 9.5 m. The conductor has a rectangular cross section (2.5 cm $\times$ 42 cm). There are 15 turns in the top half of the magnet and 15 in the bottom half, for a total length of 30 circumferences (the top and bottom coils are connected in series). If we apply 100 V to the coils, what current flows through it? How much power does this require to run? **ANSWER**: The cross-sectional area of a conductor is $A = 0.025 \times 0.42 = 0.0105$ m^2. One circumference is $2\pi \times 9.5 = 59.69$ m, so 30 circumferences is $\ell = 1790.7$ m. Thus the resistance of the combined coils is $R = \rho\ell/A = 0.004775\ \Omega$ and a voltage of 100 V will produce a current $\boxed{i = V/R = 20941 \text{ Amp}}$ for a power consumption (due to Ohmic heating) of $\boxed{P = VI = 2.094 \times 10^6 \text{ W (2.094 MW)}}$.

2. **DISTRIBUTED LOAD:** [Challenge problem!] A power transmission line (for instance) can be modelled as an array of discrete resistors such as that shown below. If the array continues indefinitely to the right, what is the effective resistance between A and B?

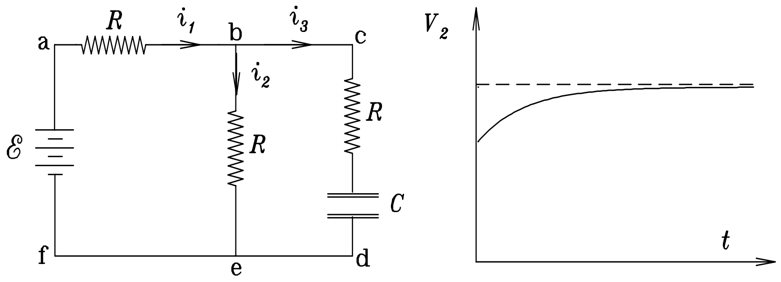


ANSWER: As in any circuit problem, you should first ask yourself, "What is going to happen?" In this case, what will happen when a voltage is applied between A and B? Well, a current I will flow down the top wire and a fraction of it will "leak off" through each of the vertical resistors across to the return wire on the bottom. Thus the current at each junction will be smaller and smaller as we move to the right, and eventually a negligible amount of current will be left; at that point we can terminate the array without noticing much difference. Therefore it should work to make a series of successive approximations in which we terminate the array further and further to the right, calculating the effective resistance of each finite array (something we know how to do), and see if the answer seems to be converging to an asymptotic value. Let's call the effective resistance R_n for the nth approximation. Using our simple rules for addition of resistances in series or parallel, we can quickly obtain $R_1 = 2R$,

$$R_2 = R + \frac{1}{\frac{1}{R} + \frac{1}{R+R}} = R + \frac{1}{\frac{1}{R} + \frac{1}{R_1}} = \frac{5}{3}R, \qquad R_3 = R + \frac{1}{\frac{1}{R} + \frac{1}{R_2}} = \frac{13}{8}R, \qquad R_4 = \frac{34}{21}R, \qquad R_5 = \frac{89}{55}R,$$

and by extension (noticing that each time we move one more to the right, we take the sum of the numerator and denominator of the previous fraction as the new denominator and set the new numerator equal to the sum of the previous numerator and the new denominator) $R_6 = \frac{233}{144}R = 1.618056R$, $R_7 = \frac{610}{377}R = 1.618037R$, $R_8 = \frac{1597}{987}R = 1.6180344R$, $R_9 = \frac{4181}{2584}R = 1.6180341R$, and so on. Obviously the result converges; but it would be nice to find a simple expression for its exact value.... If the chain is truly *infinite*, then you can chop off the first pair of resistors (one horizontal and one vertical on the sketch) and the array that remains is *the same as the one you started with*. Therefore it can be replaced by a single resistor R_∞ with the same resistance as the effective resistance of the array we set out to analyze. This converts the infinite array to something resembling R_2 in the previous solution except with the two "outer" resistors replaced by a single R_∞. The formula is then straightforward: $R_\infty = R + \dfrac{1}{\frac{1}{R} + \frac{1}{R_\infty}}$ $\longrightarrow$ [algebra] $\longrightarrow R_\infty^2 - RR_\infty - R^2 = 0$ to which one can apply the Quadratic Theorem to obtain $R_\infty = \dfrac{R \pm \sqrt{R^2 + 4R^2}}{2}$ or, discarding the unphysical result with $R_\infty < 0$, $\boxed{R_\infty = \dfrac{1+\sqrt{5}}{2}\,R = 1.61803\ldots R}$.

3. RC CIRCUIT TIME-DEPENDENCE: In the circuit shown, $\mathcal{E} = 1.2$ kV, $C = 6.5$ μF and $R_1 = R_2 = R_3 = R = 0.73$ MΩ. With C completely uncharged, switch **S** is suddenly closed (at $t = 0$).



(*a*) Determine the currents through each resistor for $t = 0$ and as $t \to \infty$. **ANSWER**: At $t = 0$, when the capacitor is completely discharged ($q = 0$), there is no voltage across C and it is effectively a "short." The circuit can thus be drawn as a simple array of resistors, giving $R_{\text{eff}} = R + [1/R + 1/R]^{-1} = R + \frac{1}{2}R = \frac{3}{2}R$ initially. The net current (which flows through R_1) is then $I_1 = \mathcal{E}/R_{\text{eff}} = 1.2 \times 10^3 \times 2/(3 \times 0.73 \times 10^6) = 1.096 \times 10^{-3}$ A or $\boxed{I_1(0) = 1.096 \text{ mA}}$. At junction **b**, I_1 splits into two equal parts, $\boxed{I_2(0) = I_3(0) = \frac{1}{2}I_1(0) = 0.548 \text{ mA}}$. As $t \to \infty$ the capacitor gets fully charged and no more current flow onto it: $\boxed{I_3(\infty) = 0}$. Then we can just ignore that part of the circuit and calculate the current through the other two resistors in series as $I_1 = I_2 = \mathcal{E}/2R = 1.2 \times 10^3/(2 \times 0.73)$ or $\boxed{I_1(\infty) = I_2(\infty) = 0.822 \text{ mA}}$.

(*b*) Draw a qualitative graph of the potential difference V_2 across R_2 as a function of time from $t = 0$ and as $t \to \infty$. **ANSWER**: We anticipate an initial voltage drop of $V_2(0) = RI_2(0)$ heading monotonically toward the final value of $V_2(\infty) = RI_2(\infty)$ and the initial change should be more rapid — this is a qualitative description of exponential decay of the difference between V_2 and its equilibrium value of $V_2(\infty)$.

(*c*) What are the numerical values of V_2 at $t = 0$ and as $t \to \infty$?
ANSWER: $\boxed{V_2(0) = 0.4 \text{ kV}}$ and $\boxed{V_2(\infty) = 0.6 \text{ kV}}$.

(*d*) Give the practical physical meaning of "as $t \to \infty$" in this case.
ANSWER: "$t \to \infty$" means $\boxed{t \gg RC = 4.745 \text{ s}}$.

(*e*) Finally, write down expressions for the currents through R_1, R_2 and R_3 as functions of time, in terms of C and R. **ANSWER**: Voltage drops around loop **abef** must sum to zero: $\mathcal{E} - I_1R - I_2R = 0$ (1). Similarly around loop **acdf**: $\mathcal{E} - I_1R - I_3R - q/C = 0$ (2). Adding (1) and (2) gives $2\mathcal{E} - 2I_1R - (I_2 + I_3)R = q/C$. But charge must be conserved at junction **b**, so $I_1 = I_2 + I_3$ (3), giving $2\mathcal{E} - 3I_1R = q/C$. Using (2) to substitute for I_1 gives $2\mathcal{E} - 3R(\mathcal{E}/R - I_3 - q/RC) = q/C$ or $I_3 \equiv dq/dt = \mathcal{E}/3R - 2q/3RC$. Taking the time derivative of both sides gives $\dfrac{dI_3}{dt} = -\dfrac{2}{3RC}I_3$ which we know has the solution $I_3(t) = I_3(0)\exp\left[-\frac{2t}{3RC}\right]$. From (*a*) we know $I_3(0) = \frac{1}{2}I_1(0) = 2\mathcal{E}/6R$, so $\boxed{I_3(t) = \dfrac{\mathcal{E}}{3R}\exp\left[-\dfrac{2t}{3RC}\right]}$. From (1) and (2) we have $I_1 = \mathcal{E}/R - (I_1 - I_3)$ or $2I_1 = \mathcal{E}/R + I_3$ giving $\boxed{I_1(t) = \dfrac{1}{2}\dfrac{\mathcal{E}}{R}\left(1 + \dfrac{1}{3}\exp\left[-\dfrac{2t}{3RC}\right]\right)}$. Finally, (1) gives $I_2 = \mathcal{E}/R - I_1$ or $\boxed{I_2(t) = \dfrac{1}{2}\dfrac{\mathcal{E}}{R}\left(1 - \dfrac{1}{3}\exp\left[-\dfrac{2t}{3RC}\right]\right)}$. For $t = 0$ and $t \to \infty$ these check with the results of (*a*) above.