THE UNIVERSITY OF BRITISH COLUMBIA

Physics 122 Assignment # 7 SOLUTIONS:
THE MAGNETIC FIELD

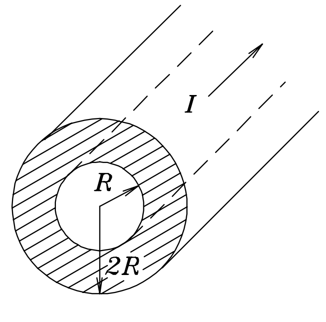
Fri. 15 Feb. 2002 — finish by Mon. 25 Feb.

1. FRICTION *vs.* THE LORENTZ FORCE: A 2.0-kg copper rod rests on two horizontal rails 2.0 m apart and carries a current of 100 A from one rail to the other. The coefficient of static friction between the rod and the rails is $\mu_s = 0.50$. What is the smallest magnetic field (not necessarily vertical) that would cause the bar to slide?
ANSWER: First let's calculate the minimum required magnetic force: the usual FBD gives $N = mg - F\sin\theta$ and $F\cos\theta = \mu N$ just before slipping. Solving for F yields $F = \mu mg/[\cos\theta + \mu\sin\theta]$. Differentiating with respect to θ gives $dF/d\theta = \mu mg(\sin\theta - \mu\cos\theta)/(\cos\theta + \mu\sin\theta)^2$, which is zero (the condition for an extremum) when $\sin\theta = \mu\cos\theta$ or when $\tan\theta = \mu = 0.50 \Longrightarrow \theta = 0.4636$. Thus $F = (0.5 \times 2 \times 9.81)/(0.8944 + 0.5 \times 0.4472) = 8.774$ N. Since the current is out of the page, we want a magnetic field $\vec{B}$ in the plane of the page at right angles to the desired force, *i.e.* making an angle $\frac{\pi}{2} + \theta = 2.034$ with the horizontal x direction. The magnitude of $B_{\min}$ is given by $F = iLB_{\min}$ with $L = 2$ m, namely $B_{\min} = F/iL = 8.774/(100 \times 2)$ or $\boxed{B_{\min} = 0.04387 \text{ T}}$.

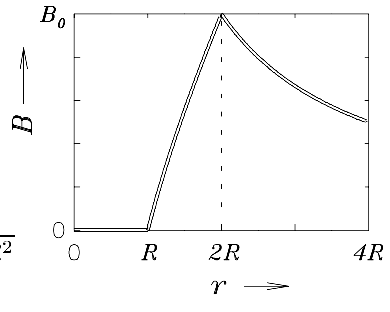
2. CYCLOTRONS: (Neglect any relativistic effects.) Suppose that we want to build a small cyclotron for protons using a magnet with a uniform field over a region 1.0 m in radius such that the protons reach a maximum kinetic energy of 20 MeV at the outer radius of the magnet. (*a*) What magnetic field must the magnet produce?
ANSWER: In each case we have $m\frac{v^2}{r} = qvB \Longrightarrow mv \equiv p = qBr$ and $\omega = \frac{v}{r} = \frac{qB}{m}$. In this case $q = e = 1.602 \times 10^{-19}$ C and $m = m_p = 1.67 \times 10^{-27}$ kg. At $K = 20$ MeV $= 20 \times 1.602 \times 10^{-19} = 3.204 \times 10^{-12}$ J, $p = \sqrt{2mK} = 1.035 \times 10^{-19}$ kg-m/s, so when $r = 1.0$ m, $B = p/qr$ gives $\boxed{B = 0.646 \text{ T}}$. (*b*) At what frequency must the "dee" voltage oscillate? **ANSWER**: $\omega = qB/m = 6.19 \times 10^7$ s^{-1} and $\boxed{f = \omega/2\pi = 9.85 \text{ MHz}}$.

Now suppose we want to build a cyclotron to accelerate electrons *without a magnet*, using the Earth's magnetic field (assume $B = 5 \times 10^{-5}$ T) to keep the electrons moving in circles. (*c*) What is the radius of the electron orbit at 100 eV? **ANSWER**: Now $m = m_e = 9.11 \times 10^{-31}$ kg, $|q| = e$, $B = 5 \times 10^{-5}$ T and $K = 100$ eV $= 1.602 \times 10^{-17}$ J $\Longrightarrow p = \sqrt{2mK} = 5.403 \times 10^{-24}$ kg-m/s giving $\boxed{r = p/qB = 0.6744 \text{ m}}$. (*d*) What is the frequency (in Hz) of the RF electric field we must supply to the cyclotron "dees?"
ANSWER: $\omega = qB/m = 8.794 \times 10^6$ s^{-1} and $\boxed{f = \omega/2\pi = 1.40 \times 10^6 \text{ Hz}}$.

3. HOLLOW CYLINDRICAL CONDUCTOR: A thick-walled hollow conducting cylinder carries a uniformly distributed current I. The (centred) hole in the middle has a radius of R and the outer radius of the conductor is $2R$. Derive an expression for the strength of the magnetic field B as a function of radial distance r from the cylinder axis, in the range from $r = R$ to $r = 2R$; then *plot* (*i.e.* sketch, showing axis labels, scales and values at key points) $B(r)$ in the range from $r = 0$ to $r = 4R$.



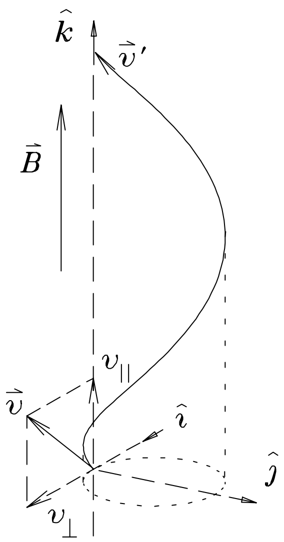
ANSWER: Throughout this problem, *symmetry* demands that the magnetic field circulate around the central axis according to the right hand rule and have the same magnitude $B(r)$ everywhere on an Ampèrian loop of radius r centred on the axis. Thus $\oint \vec{B} \cdot d\vec{s} = 2\pi r B = \mu_0 i_{\text{encl}}$ for every such loop and all there is to the calculation is to determine the current i_{encl} enclosed by the loop for each r. When $r < R$, *no* current is enclosed so $B(r < R) = 0$. For $r > 2R$, *all* the current is enclosed so $B(r > 2R) = \frac{\mu_0 I}{2\pi r}$ just as for a *wire* along the axis. The only nontrivial case is for $R < r < 2R$, where the enclosed current is given by $\frac{i_{\text{encl}}}{I} = \frac{\pi r^2 - \pi R^2}{\pi(2R)^2 - \pi R^2}$ — *i.e.* the fraction of current is the fraction of cross-sectional area.



This gives $\boxed{B(R<r<2R) = \frac{\mu_0 I}{2\pi r} \cdot \frac{(r^2 - R^2)}{3R^2}}$. At $r = 2R$, $B(2R) \equiv B_0 = \frac{\mu_0 I}{4\pi R}$. (See sketch.)

Challenge Problems:

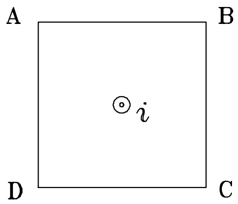
1. MOTION OF AN ELECTRON IN A MAGNETIC FIELD: An electron has a kinetic energy of 400 eV as it moves through a region containing a uniform magnetic field $\vec{B} = B\hat{k}$ of magnitude $B = 4 \times 10^{-4}$ T. At $t = 0$ it is at the origin of coordinates $(x = 0, y = 0, z = 0)$ and has velocity components $v_y = 0$ and $v_x = v_z > 0$. Find the position of the electron (x, y and z) 10 ns later. [1 ns = 10^{-9} s]



ANSWER: The kinetic energy is $K = \frac{1}{2}mv^2 = 400 \times 1.602 \times 10^{-19} = 6.41 \times 10^{-17}$ J and $m = 9.11 \times 10^{-31}$ kg, so $v = \sqrt{2K/m} = 1.186 \times 10^7$m/s. Since $\vec{v} = v_x\hat{\imath} + v_z\hat{k}$ and $v_x \equiv v_\perp = v_z \equiv v_\parallel$, we have $v_\parallel = v_\perp = v/\sqrt{2} = 8.388 \times 10^6$ m/s. Now, $\vec{B} = B\hat{k}$ so $v_\parallel$ is parallel to $\vec{B}$ and therefore continues unaffected by the magnetic field, while $v_\perp$ bends in a circle of radius $r = mv_\perp/qB = 0.1192$ m. The electron is negatively charged, so it bends to the left in an "up" field (you can verify this from the right-hand rule for $\vec{F} = q\vec{v} \times \vec{B}$) and so the path is a spiral with constant radius and pitch, as shown in the figure. The frequency of the orbit of $v_\perp$ is given by $\omega = v_\perp/r = qB/m = 7.035 \times 10^7$ s^{-1}. At $t = 10 \times 10^{-9}$ s the angle swept out by the spiral since $t = 0$ is $\theta = \omega t = 0.7035$ radians. This gives $x = r\sin\theta$ and $y = r(1 - \cos\theta)$ the values $\boxed{x = 0.0771 \text{ m}}$ and $\boxed{y = 0.0283 \text{ m}}$. Meanwhile $\boxed{z = v_\parallel t = 0.0839 \text{ m}}$.

2. FORCE ON A CURRENT-CARRYING CONDUCTOR: A long, rigid conductor, lying along the x axis, carries a current of 6.0 A in the $-\hat{\imath}$ direction. A magnetic field $\vec{B} = 3.0\hat{\imath} + 6.0x^2\hat{\jmath}$ (with x in m and B in mT) is present. Calculate the vector force on the 3.0-m segment of the conductor that lies between $x = 1.0$ m and $x = 4.0$ m. **ANSWER**: In general $d\vec{F} = i d\vec{\ell} \times \vec{B}$. In this case (suppressing the SI units) we have $i d\vec{\ell} = -6.0\hat{\imath}dx$ [Note: $\hat{\imath}$ is the unit vector in the x direction; do not confuse $\hat{\imath}$ with the current, i. Sorry, there are not enough letters!] so $d\vec{F} = dx(-6.0\hat{\imath}) \times (3.0\hat{\imath} + 6.0x^2\hat{\jmath}) \times 10^{-3} = dx(-0.036x^2)\hat{k}$ (remember, $\hat{\imath} \times \hat{\imath} = 0$) which we must then integrate over the specified range to get $\vec{F} = -0.036\hat{k}\int_{1.0}^{4.0} x^2\,dx = -0.012\hat{k}\left[x^3\right]_1^4 = -0.012\hat{k}(64 - 1)$ or $\boxed{\vec{F} = -(0.756 \text{ N})\hat{k}}$.

3. AMPÈRE'S LAW: A wire carrying a current of 2000 A coming out of the page, as shown, emerges from the centre of the square ABCD whose side is 3.0 m in length. (a) Using Ampère's Law, find the average value along AB of the magnetic field component parallel to AB. (b) Find the magnitude and direction of the magnetic field at the midpoint of the line AB.



ANSWER: (a) The magnetic field does not point directly along the Ampèrian loop everywhere, nor is it constant in magnitude along the loop; but we are only interested in the average component parallel to AB. By symmetry, whatever $\vec{B}$ does along one side it does along the other three sides as well, so $\int_A^B \vec{B} \cdot d\vec{s} \equiv (3.0 \text{ m})\langle B_\parallel \rangle_{AB} = -\frac{1}{4}\oint_{ABCD} \vec{B} \cdot d\vec{s} = \mu_0 i$, where the – sign indicates that the direction of $\vec{B}$ is generally from B to A (rather than from A to B) by the right-hand rule. All this gives $\langle B_\parallel \rangle_{AB} = \mu_0 i/3.0 = -\frac{1}{4}4\pi \times 10^{-7} \times 2000/3.0$ or $\boxed{\langle B_\parallel \rangle_{AB} = -2.094 \times 10^{-4} \text{ T}}$. ($b$) The field at the midpoint of AB points toward A (by symmetry and the right-hand rule) and has a magnitude given by the usual formula for the field due to a long straight wire, namely $\boxed{B = \frac{\mu_0 i}{2\pi r} = \frac{2 \times 10^{-7} \times 2000}{1.5} = 2.67 \times 10^{-4} \text{ T}}$.