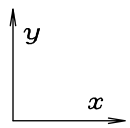
Solutions to Laplace's Equation: $\nabla^2 V = 0$

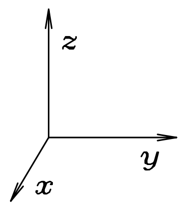
2D Cartesian:



$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} + \left.\begin{matrix} e^{kx} \\ e^{-kx} \end{matrix}\right\} \left.\begin{matrix} \cos ky \\ \sin ky \end{matrix}\right\} \quad + \text{ permutations } (x \leftrightarrow y).$$

3D Cartesian:

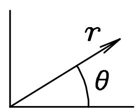


$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x,y,z) = \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} \left.\begin{matrix} z \\ 1 \end{matrix}\right\} + \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} \cos py \\ \sin py \end{matrix}\right\} \left.\begin{matrix} e^{qz} \\ e^{-qz} \end{matrix}\right\} + \left.\begin{matrix} e^{px} \\ e^{-px} \end{matrix}\right\} \left.\begin{matrix} \cos qy \\ \sin qy \end{matrix}\right\} \left.\begin{matrix} \cos \sqrt{p^2 - q^2}\, z \\ \sin \sqrt{p^2 - q^2}\, z \end{matrix}\right\}$$

$$+ \text{ all permutations } \{x, y, z\}.$$

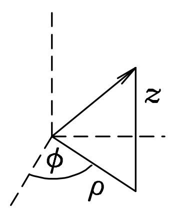
2D Plane Polar:



$$\nabla^2 V \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

$$V(r,\theta) = \left.\begin{matrix} \ln r \\ 1 \end{matrix}\right\} + \left.\begin{matrix} r^n \\ r^{-}n \end{matrix}\right\} \left.\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\}$$

3D Cylindrical:

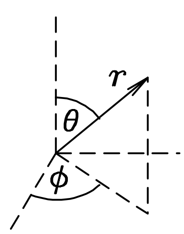


$$\nabla^2 V \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(\rho,\phi,z) = \left.\begin{matrix} J_n(k\rho) \\ N_n(k\rho) \end{matrix}\right\} \left.\begin{matrix} \cos n\phi \\ \sin n\phi \end{matrix}\right\} \left.\begin{matrix} e^{kz} \\ e^{-kz} \end{matrix}\right\}$$

where $J_n(k\rho) \to$ Bessel functions and $N_n(k\rho) \to$ Neumann functions.

3D Spherical:



$$\nabla^2 V \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$V(r,\theta,\phi) = \left.\begin{matrix} r^\ell \\ r^{-(\ell+1)} \end{matrix}\right\} \left.\begin{matrix} P_\ell^m(\cos\theta) \\ Q_\ell^m(\cos\theta) \end{matrix}\right\} \left.\begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix}\right\}$$

where $P_\ell^m(\cos\theta)$ are associated Legendre polynomials

and $Q_\ell^m(\cos\theta)$ are associated Legendre polynomials of the second kind.

If axial symmetry then $V(r,\theta,\phi) = \left.\begin{matrix} r^\ell \\ r^{-(\ell+1)} \end{matrix}\right\} \left.\begin{matrix} P_\ell(\cos\theta) \\ Q_\ell(\cos\theta) \end{matrix}\right\}$

where $P_\ell(\cos\theta)$ are Legendre polynomials and $Q_\ell(\cos\theta)$ are Legendre polynomials of the second kind.

Match **linear combinations** of the forms above to the appropriate **boundary conditions** imposed by (*e.g.*) conducting surfaces (equipotentials) and any requirements that $V \xrightarrow[r\to\infty]{} 0$ *etc.*