The University of British Columbia

Physics 401 Assignment # 5: RELATIVISTIC ELECTRODYNAMICS

SOLUTIONS:

Wed. 1 Feb. 2006 — finish by Wed. 8 Feb.

This is a relatively [pun intended] short Assignment, since the first Midterm Exam is on Monday February 6 (in class, 50 minutes). Nevertheless it will count the same as other Assignments, if you choose to tackle it. For the exam, you may bring your own *1-page summary sheet* with any hard-to-remember equations *etc*. The exam will cover all of Chapters 7 and 8, sections 10.1 of Chapter 10, and all of Chapter 12. There will not be anything on the Midterm explicitly about $F^{\mu\nu}$ (second problem below) but applications of Eqs. (12.108) are fair game.

1. If we stay in the Lorentz gauge ($\partial_\mu A^\mu = 0$), A^μ is a genuine 4-vector and therefore $A_\mu A^\mu$ is a Lorentz scalar. Use this fact to show that, in *any* frame, $|\vec{\boldsymbol{J}}|^2 = c^2(\rho^2 - \rho_0^2)$, where ρ_0 is the charge density in the frame where $\vec{\boldsymbol{J}} = 0$. **ANSWER**: If $A_\mu A^\mu$ is really a Lorentz scalar, then it has the same value in any reference frame as it does in the frame where $\vec{\boldsymbol{J}} = 0$, namely (in Griffiths' notation) $-c^2\rho_0^2$. In general, $A_\mu A^\mu = -c^2\rho^2 + |\vec{\boldsymbol{J}}|^2$. Setting these equal and rearranging terms gives the desired result. This is just a warm-up exercise in "invariant thinking".

2. Use the formal definition of the Field Tensor $F^{\mu\nu}$ in Eq. (12.118)[1] and the rule for its Lorentz transformation,[2] $F'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$, to derive the equations analogous to Eqs. (12.108) describing the transformation of $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ under a "boost" into a reference frame moving at velocity u (with the usual corresponding definitions of β and γ) in the positive $\hat{\boldsymbol{z}}$ direction.[3] **ANSWER**: This one drove me nuts until I broke the "matrix multiplication" habit I learned in High School. Here we *don't* find the $\mu\nu$th component of the product of two tensors by summing the products of the elements of the μth row of the first with the elements of the νth *column* of the second; instead we sum the products of the elements of the μth row of the first with the elements of the νth **row** of the second. Doing this twice [first with ΛF and then with $\Lambda(\Lambda F)$] gives $\Lambda F =$

$$\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & e_x & e_y & e_z \\ -e_x & 0 & B_z & -B_y \\ -e_y & -B_z & 0 & B_x \\ -e_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} -\beta\gamma e_z & (-\gamma e_x + \beta\gamma B_y) & (-\gamma e_y - \beta\gamma B_z) & -\gamma e_z \\ e_x & 0 & -B_z & B_y \\ e_y & B_z & 0 & -B_x \\ \gamma e_z & (\beta\gamma e_x - \gamma B_y) & (\beta\gamma e_y + \gamma B_x) & \beta\gamma e_z \end{pmatrix}$$

$$\text{and} \quad \Lambda(\Lambda F) = \begin{pmatrix} (-\beta\gamma^2 e_z + \beta\gamma^2 e_z) & (\gamma e_x - \beta\gamma B_y) & (\gamma e_y + \beta\gamma B_z) & (\gamma^2 e_z - \beta^2\gamma^2 e_z) \\ (-\gamma e_x + \beta\gamma B_y) & 0 & B_z & (\beta\gamma e_x - \gamma B_y) \\ (-\gamma e_y - \beta\gamma B_x) & -B_z & 0 & (\beta\gamma e_y + \gamma B_x) \\ (\beta^2\gamma^2 e_z - \gamma^2 e_z) & (-\beta\gamma e_x + \gamma B_y) & (-\beta\gamma e_y - \gamma B_x) & (-\beta\gamma^2 e_z + \beta\gamma^2 e_z) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \gamma(e_x - \beta B_y) & \gamma(e_y + \beta B_z) & e_z \\ \gamma(-e_x + \beta B_y) & 0 & B_z & \gamma(\beta e_x - B_y) \\ \gamma(-e_y - \beta B_x) & -B_z & 0 & \gamma(\beta e_y + B_x) \\ e_z & \gamma(-\beta e_x + B_y) & \gamma(-\beta e_y - B_x) & 0 \end{pmatrix}$$

where $\vec{\boldsymbol{e}} \equiv \vec{\boldsymbol{E}}/c$ has been defined for compactness.

$$\text{That is,} \quad \boxed{\begin{array}{lll} E'_x = \gamma(E_x - uB_y) & E'_y = \gamma(E_y + uB_z) & E'_z = E_z \\ B'_x = \gamma(B_x + \beta E_y/c) & B'_y = \gamma(B_y - \beta E_x/c) & B'_z = B_z \end{array}} \,.$$

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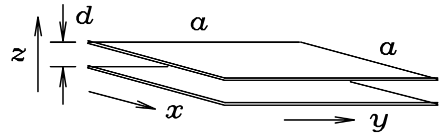
[1] Let's acquiesce to Griffiths' notation for this problem.

[2] The Einstein summation convention is assumed: sum over all repeated indices.

[3] Eqs. (12.108) are for a "boost" in the $\hat{\boldsymbol{x}}$ direction. This is rather tedious, especially since the result is rather obvious when you're done, but everyone should do it once in order to understand that the definition of $F^{\mu\nu}$ is more useful formally (for elegantly expressing the essence of electromagnetism) than for solving practical problems.

3. A capacitor made from two square parallel plates a on a side and d apart is given a charge $-Q$ on the upper plate and $+Q$ on the lower plate. Let the origin of coordinates be in the centre of the capacitor, the edges of the plates parallel to $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ and the gap in the $\hat{\boldsymbol{z}}$ direction.

Using Lorentz transformations, find $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ inside the capacitor



(*a*) in a frame moving at a velocity u in the $\hat{\boldsymbol{x}}$ direction;
ANSWER: For this we can just plug $E_x = E_y = \vec{\boldsymbol{B}} = 0$ and $E_z = \epsilon_0 Q/a^2$ into Eqs. (12.108) to get the only nonzero components $\boxed{E_z' = \gamma \epsilon_0 Q/a^2}$ and $B_y' = \gamma(u/c^2)\epsilon_0 Q/a^2$ where as usual $\gamma \equiv \left(1 - u^2/c^2\right)^{-1/2}$.

(*b*) in a frame moving at a velocity u in the $\hat{\boldsymbol{z}}$ direction. **ANSWER**: Here we can use the analogous equations derived in the previous problem to get the only nonzero component $E_z' = E_z = \epsilon_0 Q/a^2$. That is, the fields have not changed. The only difference in geometry is that the plates are closer together due to Lorentz contraction; but that does not affect the electric field, as we know.

(*c*) For the former case, compare your result with what you would expect from simply transforming the dimensions of the plates into the moving frame and treating their motion as sheets of current.
ANSWER: Lorentz contraction shrinks distances parallel to the motion, but not perpendicular. Thus the plates are no longer square; the edge in the $\hat{\boldsymbol{x}}$ direction is a factor of $1/\gamma$ shorter in the primed frame while the other edge still has a length a. This compresses the net charge Q (an invariant) into an area smaller by the same factor, increasing σ' and consequently E_z' by a factor of γ. ✓ Meanwhile the plates are now sheets of current carrying $\vec{\boldsymbol{K}}' = \mp \sigma' u \hat{\boldsymbol{x}}$ for the top and bottom plates, respectively.[4] As for the electric field, $\sigma' = \gamma Q/a^2$. We know that two opposite sheets of current produce a uniform magnetic field between them given by $\vec{\boldsymbol{B}} = (\mu_0/2)K\hat{\boldsymbol{y}}$ or $B_y = (\mu_0/2)(\sigma' u) = \gamma Q(\mu_0/2a^2)u$. Multiplying by (ϵ_0/ϵ_0) and noting that $\mu_0\epsilon_0 = 1/c^2$ gives $\boxed{B_y' = \gamma(u/c^2)\epsilon_0 Q/a^2}$. ✓

In a simple case like this, we get the right answer by simply taking Lorentz contraction into account. So one might wonder if all the elaborate formalism is ever any real help in solving practical problems. The answer is probably, "It depends on how flawlessy you can visualize all the effects without losing track of any." Equations like (12.108) allow you to transform into the frame of your choice "without much thought" even for rather complicated situations (*e.g.* when the fields vary with position). But without relativity we would have no way of making E&M obey NEWTON'S LAWS, of which we are understandably fond. It emerges naturally that magnetic fields are an obligatory extension of COULOMB'S LAW when we transform consistently into moving frames; and that knowledge is worth having!

[4] In the moving frame, the plates appear to be moving in the opposite direction; thus the current in the positive plate is in the $-\hat{\boldsymbol{x}}$ direction and that in the negative plate is in the $+\hat{\boldsymbol{x}}$ direction.