THE UNIVERSITY OF BRITISH COLUMBIA

Physics 401 Assignment # 8:
GUIDED WAVES
SOLUTIONS:

Wed. 1 Mar. 2006 — finish by Wed. 8 Mar.

1. (p. 405, Problem **9.25**) — **Group Velocity:** Assuming negligible damping ($\gamma_j \approx 0$), calculate the group velocity ($v_g \equiv d\omega/dk$) of the waves described by Eqs. (9.166) and (9.169):

$$\tilde{\boldsymbol{E}}(z,t) = \tilde{E}_0 \, e^{-\kappa z} \, e^{i(kz-\omega t)} \qquad \text{and} \qquad \tilde{k} \simeq \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right].$$

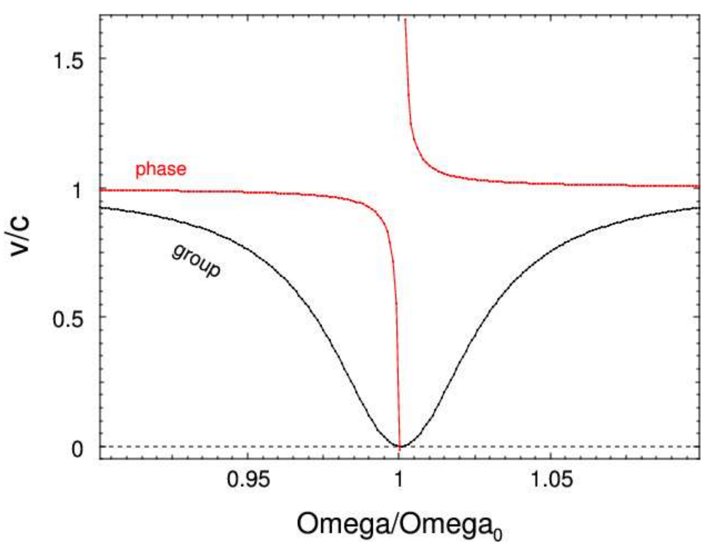
Show that $v_g < c$, even when $v > c$.

ANSWER: Setting $\gamma = 0$ makes $\tilde{k} = k(\omega)$ pure real, simplifying matters considerably. We can just invert the derivative: $v_g \equiv d\omega/dk = [dk/d\omega]^{-1}$ so first we find

$$\begin{aligned}
\frac{dk}{d\omega} &= \frac{1}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2}\right) + \frac{\omega}{c}\frac{Nq^2}{2m\epsilon_0}\sum_j f_j \frac{d}{d\omega}\left\{[\omega_j^2-\omega^2]^{-1}\right\} \\
&= \frac{1}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2} + \omega\frac{Nq^2}{2m\epsilon_0}\sum_j f_j\left[\frac{2\omega}{\left(\omega_j^2-\omega^2\right)^2}\right]\right) = \frac{1}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum_j f_j\left[\frac{1}{\omega_j^2-\omega^2} + \frac{2\omega^2}{\left(\omega_j^2-\omega^2\right)^2}\right]\right) \\
&= \frac{1}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum_j f_j\left[\frac{\omega_j^2-\omega^2+2\omega^2}{\left(\omega_j^2-\omega^2\right)^2}\right]\right) = \frac{1}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum_j f_j\left[\frac{\omega_j^2+\omega^2}{\left(\omega_j^2-\omega^2\right)^2}\right]\right)
\end{aligned}$$

from which it follows that $$\boxed{v_g = c\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{(\omega_j^2+\omega^2)f_j}{\left(\omega_j^2-\omega^2\right)^2}\right]^{-1}}.$$

The following graph was produced numerically from Eq. (9.170) with $\gamma = 0$ and only one resonant species:



The same sort of thing happens at each resonance. Since $v_g = \frac{c}{1+X}$ and X is always positive, $v_g < c$. The same is not true for $v_{\rm ph} \equiv \frac{\omega}{k}$ since the corresponding X factor can be positive or negative depending on whether ω is smaller or larger than a given ω_j.

2. (p. 411, Problem **9.27**) — **No TE$_{00}$:** Show that the mode TE$_{00}$ cannot occur in a rectangular wave guide. [*Hint:* In this case $\omega/c = k$, so Eqs. (9.180) are indeterminate, and you must go back to Eqs. (9.179). Show that B_z is a constant, and hence — applying Faraday's law in integral form to a cross section — that $B_z = 0$, so this would be a TEM mode.]

ANSWER: From the definition of the mode numbers m and n, a TE$_{00}$ mode would have $k_x = k_y = 0$ — *i.e.* we'd have an ordinary plane wave propagating straight down the waveguide at c. Neither $\vec{\boldsymbol{E}}$ nor $\vec{\boldsymbol{B}}$ would depend on x or y within the guide, only on z and t. Both would be perpendicular to $\hat{\boldsymbol{z}} = \hat{\boldsymbol{k}}$; like the sides of the guide, one would be parallel to $\hat{\boldsymbol{x}}$ and the other would be parallel to $\hat{\boldsymbol{y}}$. Thinking in these terms, we can see several reasons why such a wave is impossible. The most obvious is that if $\vec{\boldsymbol{E}}_\parallel$ does not depend on x or y, there is no way for it to change continuously to zero inside the conductor, as it must. The same applies to $B_\perp$. However, we are not encouraged by Griffiths to visualize the wave as a superposition of reflections (or, in this case, *no* reflections) of a simple plane wave; in principle there might be some exotic (x, y) dependence that would keep $k_x = k_y = 0$ and still satisfy MAXWELL'S EQUATIONS, so we go the formal route and prove otherwise.

For any TE wave, $E_z = 0$ by definition. Equations (9.179) then become

$$
\begin{array}{rll}
(i) & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z & \\
(ii) & -ikE_y = i\omega B_x & \Rightarrow E_y = -cB_x \\
(iii) & ikE_x = i\omega B_y & \Rightarrow E_x = cB_y \\
(iv) & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 & \Rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \\
(v) & \frac{\partial B_z}{\partial y} - ikB_y = -i\frac{k}{c}E_x & \Rightarrow \frac{\partial B_z}{\partial y} = 0 \\
(vi) & ikB_x - \frac{\partial B_z}{\partial x} = -i\frac{k}{c}E_y & \Rightarrow \frac{\partial B_z}{\partial x} = 0
\end{array}
$$

The last two results show that B_z is independent of x or y, *i.e.* **constant** over any x-y plane. Thus if we apply the integral form of FARADAY'S LAW over a loop in any x-y plane just inside the metal of the guide (where $E = 0$), we get $0 = -ab\frac{\partial B_z}{\partial t} = i\omega ab B_z$ $\Rightarrow B_z = 0$, *i.e.* the wave is also a TM wave, so it is a TEM mode, which was shown at the bottom of p. 407 to be impossible in a hollow waveguide. ✓ $\mathcal{QED}$

3. (p. 411, Problem **9.28**) — **TE Modes:** Consider a rectangular wave guide with dimensions 2.28 cm × 1.01 cm. What TE modes will propagate in this waveguide, if the driving frequency is 1.70×10^{10} Hz? Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use? What are the corresponding wavelengths (in open space)? **ANSWER**: Our driving frequency is $\omega = 2\pi\nu = 1.068 \times 10^{11}$ s^{-1}. If $a = 2.28 \times 10^-2$ m and $b = 1.01 \times 10^-2$ m, $\omega_{mn} \equiv \pi c\sqrt{(m/a)^2 + (n/b)^2}$ $= \sqrt{m^2(0.4131)^2 + n^2(0.9325)^2} \times 10^{11}$ s^{-1}. The corresponding modes will propagate only if $\omega > \omega_{mn}$:

$$
\begin{array}{rcll}
\omega_{10} & = & 0.4131 \times 10^{11}\ \mathrm{s}^{-1} & \text{OK} \\
\omega_{20} & = & 0.8262 \times 10^{11}\ \mathrm{s}^{-1} & \text{OK} \\
\omega_{30} & = & 1.2392 \times 10^{11}\ \mathrm{s}^{-1} & \text{NO} \\
\omega_{01} & = & 0.9325 \times 10^{11}\ \mathrm{s}^{-1} & \text{OK} \\
\omega_{02} & = & 1.8650 \times 10^{11}\ \mathrm{s}^{-1} & \text{NO} \\
\omega_{11} & = & 1.0199 \times 10^{11}\ \mathrm{s}^{-1} & \text{OK}
\end{array}
$$

And clearly no higher modes will propagate at that frequency. If we wanted only a single TE mode to propagate, it would be the TE$_{10}$ (lowest) mode, for which the allowable frequency range would be from 0.4131×10^{11} to 0.8262×10^{11} s^{-1} or $\boxed{\nu = 6.5744 \text{ to } 13.1488 \text{ GHz}}$, corresponding to $\boxed{\lambda = c/\nu = 4.56 \text{ down to } 2.28 \text{ cm}}$.

4. (p. 411, Problem **9.30**) — **TM Modes:** Work out the theory of TM modes for a rectangular wave guide. In particular, find the longitudinal electric field, the cutoff frequencies, and the wave and group velocities. Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given wave guide. [*Caution:* What is the lowest TM mode?] **ANSWER**: TM mode means $B_z = 0$, so only the first of Eqs. (9.181) is needed: $\left(\nabla_T^2 + k_T^2\right) E_z = 0$ where $\nabla_T^2 =\equiv \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^2}$ and $k_T^2 \equiv (\omega/c)^2 - k^2$. This is known as the HELMHOLTZ EQUATION in two dimensions, and is solved by separation of variables just as on pp. 408-409 except with one twist: instead of demanding $B_x = 0$ at $x = 0$ and $x = a$ and $B_y = 0$ at $y = 0$ and $y = b$ (to satisfy the continuity of $B_\perp$) we must have $E_z = 0$ at all surfaces in order to satisfy the continuity of $\vec{\boldsymbol{E}}_\parallel$. Since $E_z(x, y) = X(x)Y(y)$, this means $X(0) = X(a) = 0$ and $Y(0) = Y(b) = 0$, so only the sin terms survive: $\boxed{E_z(x, y) = E_0 \sin(k_x x)\sin(k_y y)}$ where $\boxed{k_x = m\pi/a \text{ and } k_y = n\pi/b}$; $\boxed{m \neq 0 \text{ and } n \neq 0}$ since either would force $E_z = 0$. Thus the first allowed TM mode is TM$_{11}$. As usual we define the cutoff frequency for the TM$_{mn}$ mode, $\boxed{\omega_{mn} \equiv c\pi\sqrt{(m/a)^2 + (n/b)^2}}$, and the dispersion relation, $\boxed{ck = \sqrt{\omega^2 - \omega_{mn}^2}}$, from which it is obvious that unless $\omega > \omega_{mn}$ the wave will not propagate (the oscillation changes to exponential decay). The wave (phase) velocity is $\boxed{v_{\text{ph}} \equiv \omega/k = c\left[1 - (\omega_{mn}/\omega)^2\right]^{-1/2}}$ and the group velocity is $v_g \equiv d\omega/dk = [dk/d\omega]^{-1}$ or $\boxed{v_g = c\left[1 - (\omega_{mn}/\omega)^2\right]^{1/2}}$ just as for TE modes. The lowest TM cutoff frequency is ω_{11}, whereas for TE it is ω_{10}, assuming $a > b$. Their ratio is $\boxed{\omega_{11}/\omega_{10} = \sqrt{1 + (a/b)^2}}$.