The University of British Columbia

Physics 401 Assignment # 9:

LOOSE ENDS & COAX CABLES

SOLUTIONS:

Wed. 8 Mar. 2006 — finish by Wed. 15 Mar.

1. **Cutoff Frequencies:** *Explain in words* why there is a *lower* limit on the frequencies of EM waves that will propagate freely either through a *tenuous plasma* or down a *rectangular waveguide*. Why is there *no* cutoff frequency (neither upper nor lower) for wave propagation down an ideal *coaxial cable*?

ANSWER: This question can be challenging if you seriously attempt to avoid formulae and explain *in words*. In fact, unless you really understand what the formulae *mean*, it's virtually impossible! Hence it is a perfect test of comprehension. Those who simply write down the relevant formulae get no more than half credit even if they are the "right" formulae. On the other hand, there are many ways of explaining in words; you get full credit for any that are not actually wrong.

First let's talk about a **plasma**, *e.g.* the ionosphere. Ions in a plasma are free to move, and so a constant electric field would produce a steady current. For "almost constant" electric fields (*i.e.* low frequency waves) the ionosphere might as well be a sheet of copper around the Earth, and like any good conductor it *reflects* electromagnetic waves — *i.e.* they do not propagate through it. However, as you turn up the frequency the ions can't move very far before the field has reversed, so the maximum amplitude of their motion gets smaller and smaller until all they do is jiggle a little as the wave propagates past them without difficulty. Also, for a given number of charged particles per unit volume, the distance between idividual charges is fixed. If the wavelength becomes short compared with that distance, one particle may go "left" while the next one goes "right" and they can no longer act "in concert" as a current density. So short wavelengths (high frequencies) "see" the plasma as a bunch of uncorrelated charges rather than a "conductor" and can propagate freely through it.

A **hollow waveguide** won't support TEM modes, so the "nice" propagation of a simple plane wave down the tube isn't allowed; we must have a wave *reflecting off the sides of the tube* and interfering with itself to produce *standing waves* with *nodes at the surfaces*. This imposes a constraint on the *wavelength* — it can't be *longer* than twice the width of the tube. And since the wave (viewed this way, as a plane wave "bouncing around") always propagates at $c = \lambda\nu$, an upper limit on wavelength implies a *lower* limit on *frequency*. $\mathcal{QED}$

The **coaxial cable** (or any other transmission line with two separate conductors that can be oppositely charged and carry opposite currents locally) *will* support TEM waves, which are basically localized plane waves propagating down the line without dispersion. They all travel at the same speed regardless of frequency and there are no restrictions on wavelength. "Anything goes."

2. **Rectangular Waveguide with Dielectric:** Show that if a hollow rectangular waveguide of the type shown in Griffiths Figure 9.24 is completely filled with a dielectric of permeability ϵ, its cut-off frequency is *lower* than if it were empty, by a factor of $\sqrt{\epsilon_0/\epsilon}$:

$$\frac{\omega_{mn}^{\text{dielectric}}}{\omega_{mn}^{\text{vacuum}}} = \sqrt{\frac{\epsilon_0}{\epsilon}} \; .$$

So, for a given operating frequency, a dielectric filled waveguide can be *smaller* than an empty one.

ANSWER: Going back to the simple model of a plane wave reflecting off the walls of the waveguide to form standing waves, nothing has changed when the guide is filled with dielectric except that the original plane wave has a lower phase velocity $v = 1/\sqrt{\epsilon\mu_0}$ instead of $c = 1/\sqrt{\epsilon_0\mu_0}$. The transverse wavevector components must still satisfy $k_x = m\pi/a$ and $k_y = n\pi/b$, so $\omega^2/v^2 = k_x^2 + k_y^2 + k_z^2$ where $k_z = k$, or $vk = \sqrt{\omega^2 - \omega_{mn}^2}$ where $\omega_{mn} \equiv \pi v\sqrt{(m/a)^2 + (n/b)^2}$. This differs from ω_{mn} for an empty waveguide only by the factor $\boxed{v/c = \sqrt{\epsilon_0/\epsilon}}$. $\mathcal{QED}$

3. (p. 412, Problem **9.31**) — **Coax Cable:**

(*a*) Show directly that Eqs. (9.197)

$$\vec{\boldsymbol{E}}(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{s}\,\hat{\boldsymbol{s}}$$
$$\vec{\boldsymbol{B}}(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{cs}\,\hat{\boldsymbol{\phi}}$$

satisfy MAXWELL'S EQUATIONS (9.177)

$$(i)\ \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}}=0 \qquad (iii)\ \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}=-\frac{\partial\vec{\boldsymbol{B}}}{\partial t}$$
$$(ii)\ \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}=0 \qquad (iv)\ \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}=\frac{1}{c^2}\frac{\partial\vec{\boldsymbol{E}}}{\partial t}$$

and the boundary conditions (9.175)

$$(i)\ \vec{\boldsymbol{E}}^{\parallel}=0 \qquad (ii)\ B^{\perp}=0\,.$$

ANSWER: We have (i) $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \frac{1}{s}\frac{\partial}{\partial s}\left[\cancel{s}\,\frac{A\cos(kz-\omega t)}{\cancel{s}}\right]=0$ ✓

(ii) $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = \frac{1}{s}\frac{\partial}{\partial\phi}\left[\frac{A\cos(kz-\omega t)}{cs}\right]=0$ ✓

(iii) $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = \frac{\partial}{\partial z}\left[\frac{A\cos(kz-\omega t)}{s}\right]\hat{\boldsymbol{\phi}}$
$= -k\frac{A\sin(kz-\omega t)}{s}\hat{\boldsymbol{\phi}}$
$= -\frac{\omega}{c}\frac{A\sin(kz-\omega t)}{s}\hat{\boldsymbol{\phi}} = -\frac{\partial\vec{\boldsymbol{B}}}{\partial t}$ ✓

and (iv) $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = -\frac{\partial B_\phi}{\partial z}\hat{\boldsymbol{s}}+\frac{1}{s}\frac{\partial\, sB_\phi}{\partial s}\hat{\boldsymbol{z}}$
$= -\frac{\partial}{\partial z}\left[\frac{A\cos(kz-\omega t)}{cs}\right]\hat{\boldsymbol{s}}$
$+\frac{1}{s}\frac{\partial}{\partial s}\left[\frac{A\ \cancel{s}\ \cos(kz-\omega t)}{c\ \cancel{s}}\right]\hat{\boldsymbol{z}}$
$= \frac{k}{c}\left[\frac{A\sin(kz-\omega t)}{s}\right]\hat{\boldsymbol{s}}$
$= \frac{\omega}{c^2}\left[\frac{A\sin(kz-\omega t)}{s}\right]\hat{\boldsymbol{s}} = \frac{1}{c^2}\frac{\partial\vec{\boldsymbol{E}}}{\partial t}$ ✓

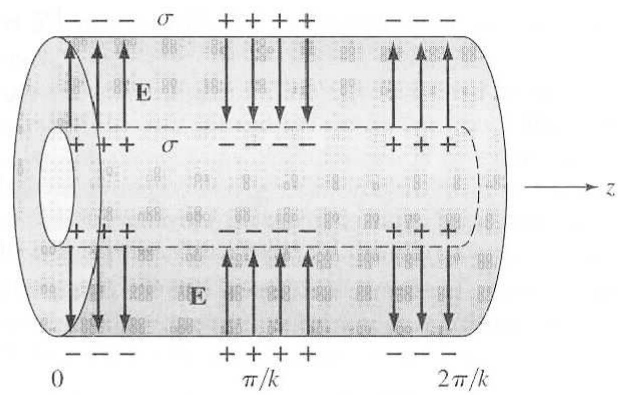
Now for the B.C.: $\vec{\boldsymbol{E}}^{\parallel}=0$ by inspection, since $\vec{\boldsymbol{E}}$ is strictly radial in direction. ✓ Similarly, since $\vec{\boldsymbol{B}}=B\hat{\boldsymbol{\phi}}$ (strictly parallel to both surfaces), $B^{\perp}=0$ as well. ✓

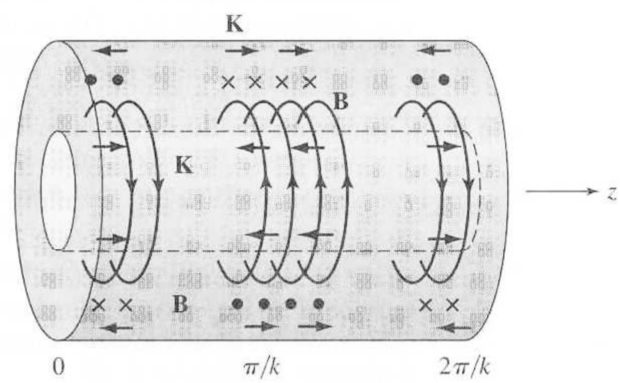
(*b*) Find the net charge per unit length, $\lambda(z,t)$, and the net current, $I(z,t)$, on the inner conductor.

ANSWER: For the charge we do GAUSS' LAW (in integral form) on a short cylindrical surface between the conductors, getting the familiar result $E=\lambda/(2\pi\epsilon_0 s)$. Thus $\lambda = 2\pi\epsilon_0 sE$ or $\boxed{\lambda = 2\pi\epsilon_0 A\cos(kz-\omega t)}$.

For the current we use AMPÈRE'S LAW (in integral form) for a circular loop between the conductors to give the familiar result $B=\mu_0 I/(2\pi s)$, which we invert to get $I = 2\pi sB/\mu_0$ or $\boxed{I=(2\pi/\mu_0 c)A\cos(kz-\omega t)}$.

Note that since $\vec{\boldsymbol{E}}=\vec{\boldsymbol{B}}=0$ outside the cable, the same two LAWs require that the charge and current on the *outer* conductor be equal and opposite to those on the inner conductor.





4. **Coax Impedance:** In class, we derived the electric and magnetic fields in a coaxial transmission line. From those we deduced the characteristic impedance of a coaxial cable:

$$Z=\frac{V(z,t)}{I(z,t)}=\frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu}{\epsilon}}=60\ \Omega\cdot\ln(b/a)$$

where a is the radius of the inner coax line and b is the radius of the outer coax cylinder, as shown above.

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In general, the characteristic impedance of a transmission line is given by

$Z = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}$, where $\mathcal{L}$ and $\mathcal{C}$ are the inductance and capacitance per unit length, respectively.

Show that the characteristic impedance of this coax line satisfies this definition by calculating $\mathcal{L}$ and $\mathcal{C}$ explicitly, and then Z.

ANSWER: The magnetic field between the conductors is $\vec{B} = \mu_0 I \hat{\phi}/(2\pi s)$ and there is no magnetic field elsewhere. The magnetic flux between the conductors in a short section of length ℓ is thus $\Phi_m = \int_a^b B(s)\ell ds$ $= (\ell\mu_0 I/2\pi)\ln(b/a)$ and so the inductance per unit length is $\boxed{\mathcal{L} \equiv \Phi_m/I\ell = \mu_0 \ln(b/a)/2\pi}$.

Similarly, a short length ℓ of the cable with equal and opposite charges $Q = \lambda\ell$ on the inner and outer conductors has an electric field strength $E(s) = \lambda/2\pi\epsilon_0 s$ at radius s and thus a resultant voltage difference $V = -\int_a^b E(s)ds$ $= (Q/2\pi\epsilon_0\ell)\ln(b/a)$. Thus the capacitance per unit length is $\boxed{\mathcal{C} \equiv Q/V\ell = 2\pi\epsilon_0/\ln(b/a)}$.

Putting the two together gives

$$\sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \sqrt{\frac{\mu_0 \ln(b/a)/2\pi}{2\pi\epsilon_0/\ln(b/a)}} = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}},$$

in agreement with the previous result for Z. ✓