THE UNIVERSITY OF BRITISH COLUMBIA

Physics 401 Assignment # 10:

RETARDED POTENTIALS

SOLUTIONS:

Wed. 15 Mar. 2006 — finish by Wed. 22 Mar.

1. (p. 426, Problem **10.8**) — **Retarded Gauge:** Confirm that the RETARDED POTENTIALS satisfy the LORENTZ GAUGE condition,

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \quad \text{or} \quad \frac{\partial A^\mu}{\partial x^\mu} = 0 \tag{1}$$

$$\text{where} \quad A^0 \equiv \frac{V}{c} \quad \left(\text{and} \quad J^0 \equiv c\rho\right) . \tag{2}$$

ANSWER: Following the *hint*, we first show

$$\vec{\nabla}\cdot\left(\frac{\vec{J}}{\mathcal{R}}\right) = \frac{1}{\mathcal{R}}(\vec{\nabla}\cdot\vec{J}) + \frac{1}{\mathcal{R}}(\vec{\nabla}'\cdot\vec{J}) - \vec{\nabla}'\cdot\left(\frac{\vec{J}}{\mathcal{R}}\right) \tag{3}$$

where $\vec{\boldsymbol{\mathcal{R}}} \equiv \vec{r} - \vec{r}'$, $\vec{\nabla}$ denotes derivatives with respect to $\vec{r}$, and $\vec{\nabla}'$ denotes derivatives with respect to $\vec{r}'$: The identity

$$\vec{\nabla}\cdot(f\vec{v}) = f\left(\vec{\nabla}\cdot\vec{v}\right) + \vec{v}\cdot\vec{\nabla} f \tag{4}$$

and the (hopefully by now familiar) results

$$\vec{\nabla}\left(\frac{1}{\mathcal{R}}\right) = -\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2} = -\vec{\nabla}'\left(\frac{1}{\mathcal{R}}\right) \tag{5}$$

$$\Longrightarrow \vec{\nabla}\cdot\left(\frac{\vec{J}}{\mathcal{R}}\right) = \frac{1}{\mathcal{R}}\left(\vec{\nabla}\cdot\vec{J}\right) - \vec{J}\cdot\left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}\right) \tag{6}$$

$$\& \quad \vec{\nabla}'\cdot\left(\frac{\vec{J}}{\mathcal{R}}\right) = \frac{1}{\mathcal{R}}\left(\vec{\nabla}'\cdot\vec{J}\right) + \vec{J}\cdot\left(\frac{\hat{\boldsymbol{\mathcal{R}}}}{\mathcal{R}^2}\right) . \tag{7}$$

Adding together Eqs. (6) and (7) gives Eq. (3). ✓
Next, noting that $\vec{J}\left(\vec{r}', t - \mathcal{R}/c\right)$ depends on $\vec{r}'$ both explicitly and through $\mathcal{R}$, whereas it depends on $\vec{r}$ only through $\mathcal{R}$, we confirm that

$$\vec{\nabla}\cdot\vec{J} = -\frac{1}{c}\dot{\boldsymbol{J}}\cdot\left(\vec{\nabla}\mathcal{R}\right) \tag{8}$$

$$\vec{\nabla}'\cdot\vec{J} = -\dot{\rho} - \frac{1}{c}\dot{\boldsymbol{J}}\cdot\left(\vec{\nabla}'\mathcal{R}\right) \ : \tag{9}$$

Derivatives of $\vec{J}(\vec{r}', t_r)$ with respect to $\vec{r}$ (on which it does not depend explicitly) mix in the *time* derivative through the *implicit* dependence of t_r on $\vec{r} = \vec{\boldsymbol{\mathcal{R}}} + \vec{r}'$. That is,

$$\vec{\nabla}\cdot\vec{J}(\vec{r}', t_r) = \left(\frac{\partial\vec{J}}{\partial t_r}\right)\cdot\vec{\nabla} t_r = -\frac{\dot{\boldsymbol{J}}\cdot\hat{\boldsymbol{\mathcal{R}}}}{c} \tag{10}$$

because, for a given $\mathcal{R}$, $\dfrac{\partial\vec{J}}{\partial t_r} = \dfrac{\partial\vec{J}}{\partial t} = \dot{\boldsymbol{J}}$, and $\vec{\nabla} t_r = -\dfrac{1}{c}\vec{\nabla}\mathcal{R} = -\dfrac{\hat{\boldsymbol{\mathcal{R}}}}{c}$. ✓
However, $\vec{J}(\vec{r}', t_r)$ depends *explicitly and implicitly* upon $\vec{r}'$, and must locally satisfy the EQUATION OF CONTINUITY $\vec{\nabla}'\cdot J = -\dot{\rho}$ (*i.e.* charge conservation) at any instant of time in terms of the source coordinates $\vec{r}'$, so we have

$$\vec{\nabla}'\cdot J(\vec{r}', t_r) = -\dot{\rho} + \frac{\dot{\boldsymbol{J}}\cdot\hat{\boldsymbol{\mathcal{R}}}}{c} \tag{11}$$

because $\vec{\nabla}' t_r = -\dfrac{1}{c}\vec{\nabla}'\mathcal{R} = +\dfrac{\hat{\boldsymbol{\mathcal{R}}}}{c}$. ✓
Finally we use this to calculate the divergence of $\vec{A}$ in Eq. (10.19):

$$A^\mu(\vec{r}, t) = \frac{\mu_0}{4\pi}\iiint\frac{J^\mu(\vec{r}', t_r)d\tau'}{\mathcal{R}} \tag{12}$$

$$\begin{aligned}\vec{\nabla}\cdot\vec{A} \;=\; \frac{\mu_0}{4\pi}\iiint\Bigg[&\frac{1}{\mathcal{R}}\left(-\frac{1}{c}\dot{\boldsymbol{J}}\cdot\hat{\boldsymbol{\mathcal{R}}}\right) \\ &+\frac{1}{\mathcal{R}}\left(-\dot{\rho} + \frac{1}{c}\dot{\boldsymbol{J}}\cdot\hat{\boldsymbol{\mathcal{R}}}\right) \\ &-\vec{\nabla}'\cdot\left(\frac{\vec{J}}{\mathcal{R}}\right)\Bigg]d\tau' .\end{aligned}$$

The DIVERGENCE THEOREM tells us that

$$\iiint\left[\vec{\nabla}'\cdot\left(\frac{\vec{J}}{\mathcal{R}}\right)\right]d\tau' = \oiint\frac{\vec{J}\cdot d\vec{a}'}{\mathcal{R}} .$$

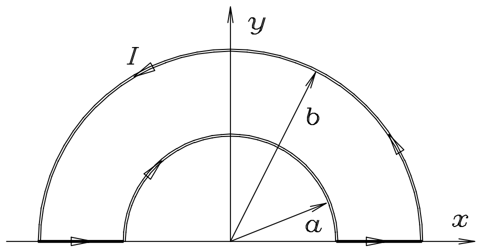
Now, if the closed surface encloses *all* the charges and currents in the source volume, $\vec{J} = 0$ over the whole surface and the surface integral is zero, leaving

$$\vec{\nabla}\cdot\vec{A} = \frac{\mu_0}{4\pi}\iiint\left(\frac{-\dot{\rho}}{\mathcal{R}}\right)d\tau'$$

$$= -\mu_0\epsilon_0\frac{\partial}{\partial t}\left\{\frac{1}{4\pi\epsilon_0}\iiint\left(\frac{\rho}{\mathcal{R}}\right)d\tau'\right\}$$

or $\boxed{\vec{\nabla}\cdot\vec{A} = -\dfrac{1}{c^2}\dfrac{\partial V}{\partial t}}$. ✓ $\mathcal{QED}$

2. (p. 427, Problem **10.10**) — **Weird Loop:**



A piece of wire bent into a weirdly shaped loop, as shown in the diagram, carries a current that increases linearly with time:

$$I(t) = kt \; .$$

(a) Calculate the retarded vector potential $\vec{A}$ at the center. **ANSWER**: Choose the origin at the same place as the field point: the centre. Thus $\vec{r} = 0$ and $\vec{\mathscr{R}} = -\vec{r}'$. The source region is uncharged, so $V = 0$.

$$\vec{A}(0,t) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}(\vec{r}', t - r'/c)}{-r'} d\ell'$$

$$= -\frac{\mu_0 k}{4\pi}\left[2\int_a^b \frac{(t-\ell/c)\hat{x}d\ell}{\ell} + \int_0^\pi \frac{(t-b/c)\hat{\theta}bd\theta}{b} - \int_0^\pi \frac{(t-a/c)\hat{\theta}ad\theta}{a}\right]$$

where $\hat{\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta$. Now, by symmetry there is as much current going "up" as "down" at the same r' and t_r, so the $\hat{y}$ components cancel. This leaves

$$\vec{A}(0,t) = \frac{\mu_0 k}{4\pi}\mathcal{I}\hat{x}$$

where

$$\mathcal{I} \equiv 2t\int_a^b \frac{d\ell}{\ell} - \frac{2}{c}\int_a^b d\ell - \left(t - \frac{b}{c}\right)\int_0^\pi \sin\theta d\theta + \left(t - \frac{a}{c}\right)\int_0^\pi \sin\theta d\theta$$

$$= 2t\ln\left(\frac{b}{a}\right) - \frac{2(b-a)}{c} - 2t + 2\frac{b}{c} + 2t - 2\frac{a}{c}$$

or $\boxed{\vec{A}(0,t) = t\frac{\mu_0 k}{2\pi}\ln\left(\frac{b}{a}\right)\hat{x}}$.

(b) Find the electric field at the center.
ANSWER: Since $V = 0$ we have just

$$\boxed{\vec{E} = -\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 k}{2\pi}\ln\left(\frac{b}{a}\right)\hat{x}} \; .$$

(c) Why does this (neutral) wire produce an *electric* field? **ANSWER**: Because the vector potential is changing with time, "Doh!" I think this is meant as a retroactive hint in case you got hung up on the preceding question.

(d) Why can't you determine the *magnetic* field from this expression for $\vec{A}$?
ANSWER: Finding $\vec{B} = \vec{\nabla}\times\vec{A}$ requires knowledge of the dependence of $\vec{A}$ on $\vec{r}$; but we have calculated $\vec{A}$ only at one point in space! If you want a differentiable $\vec{A}(\vec{r})$ you will have a far more difficult calculation to perform.

3. (p. 434, Problem **10.13**) — **Circulating Charge:** A particle of charge q moves in a circle of radius a at constant angular velocity ω. [Assume that the circle lies in the xy plane, centered at the origin, and that at time $t = 0$ the charge is at $(a, 0)$, on the positive x axis.] Find the LIÉNARD-WIECHERT POTENTIALS for points on the z axis. **ANSWER**: In general,

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{c}{\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}}\right]_{\text{ret}}$$

$$\vec{A}(\vec{r},t) = V(\vec{r},t)\left[\frac{\vec{v}}{c^2}\right]_{\text{ret}}$$

where $[\cdots]_{\text{ret}}$ means that the quantities in the square brackets are to be evaluated at the retarded time $t_r = t - \mathscr{R}/c$. Relative to the origin, $\vec{r}' = a\hat{s} = a\left[\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right]$.
For a point on the z axis, $\vec{r} = z\hat{z}$ and $\vec{\mathscr{R}} = z\hat{z} - a\cos(\omega t)\hat{x} - a\sin(\omega t)\hat{y}$ so $\mathscr{R} = \sqrt{z^2 + a^2}$, independent of time. We also have $\vec{v} = a\omega\left[-\hat{x}\sin(\omega t) + \hat{y}\cos(\omega t)\right]$ and $v = a\omega$. Thus $\vec{\mathscr{R}}(t_r) = z\hat{z} - a\cos\theta_r\hat{x} - a\sin\theta_r\hat{y}$ and $\vec{v}(t_r) = a\omega\left[-\hat{x}\sin\theta_r + \hat{y}\cos\theta_r\right]$ where $\boxed{\theta_r \equiv \omega(t - \sqrt{z^2+a^2}/c)}$. Then $\vec{\mathscr{R}}(t_r)\cdot\vec{v}(t_r) = a^2\omega[\cos\theta_r\sin\theta_r - \sin\theta_r\cos\theta_r] = 0$, leaving

$$\boxed{V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+a^2}}} \quad \text{and}$$

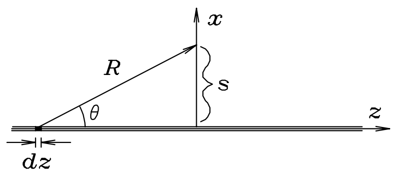
$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{a\omega q}{\sqrt{z^2+a^2}}\left[-\hat{x}\sin\theta_r + \hat{y}\cos\theta_r\right]} \; .$$

4. (p. 441, Problem **10.19**) — **Sliding String of Charges:** An infinite, straight, uniformly charged string, with λ charge per unit length, slides along parallel to its length at a constant speed v.

(a) Calculate the electric field a distance d from the string, using Eq. (10.68):

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t) = \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\boldsymbol{R}}}{R^2}$$

where $\vec{\boldsymbol{R}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{v}}t$.



ANSWER: Suppose the field point is a perpendicular distance s from the string; measure z from the nearest point on the string, as shown in the diagram. Equation (10.68), in which we do *not* need to evaluate anything at a retarded time, gives the contribution to $\vec{\boldsymbol{E}}$ from a single charge q. We need to superimpose such contributions from all charge elements $dq = \lambda dz$ at positions $-\infty < z < +\infty$ down the string: for each of these we use $\vec{\boldsymbol{R}} = s\hat{\boldsymbol{x}} - z\hat{\boldsymbol{z}}$:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t) = \frac{\lambda}{4\pi\epsilon_0}(1-v^2/c^2)\,\vec{\boldsymbol{\mathcal{I}}} \qquad \text{where}$$

$$\vec{\boldsymbol{\mathcal{I}}} \equiv \int_{-\infty}^{\infty}\frac{\hat{\boldsymbol{R}}}{R^2}\frac{dz}{\left(1-\beta^2\sin^2\theta\right)^{3/2}} .$$

For each element dz at z there is an equal element dz at $-z$; thus the "horizontal" components cancel, leaving only the x component of $\hat{\boldsymbol{R}}$, namely $\hat{\boldsymbol{x}}\sin\theta$. Meanwhile, since $s = R\sin\theta$, $1/R^2 = \sin^2\theta/s^2$; and since $-z = s\cot\theta$, $dz = s\csc^2\theta d\theta = s d\theta/\sin^2\theta$. So $dz/R^2 = d\theta/s$ and

$$\vec{\boldsymbol{\mathcal{I}}} = \int_0^{\pi}\frac{\hat{\boldsymbol{x}}\sin\theta d\theta}{s\left(1-\beta^2\sin^2\theta\right)^{3/2}} .$$

Let $u = \cos\theta$ so that $\sin\theta d\theta = -du$ and $\sin^2\theta = 1-u^2$:

$$\vec{\boldsymbol{\mathcal{I}}} = \frac{\hat{\boldsymbol{x}}}{s}\int_{-1}^{1}\frac{du}{\left(1-\beta^2[1-u^2]\right)^{3/2}}$$

$$= \frac{\hat{\boldsymbol{x}}}{s\beta^3}\int_{-1}^{1}\frac{du}{\left(a^2+u^2\right)^{3/2}}$$

$$= \frac{\hat{\boldsymbol{x}}}{s\beta^3}\left[\frac{u}{a^2\sqrt{a^2+u^2}}\right]_{-1}^{1}$$

where $\quad a^2 \equiv \frac{1}{\beta^2} - 1 . \quad$ Thus

$$\vec{\boldsymbol{\mathcal{I}}} = \frac{\hat{\boldsymbol{x}}}{s\beta^3}\left[\frac{2}{\left(\frac{1}{\beta^2}-1\right)\sqrt{\frac{1}{\beta^2}-1+1}}\right]$$

$$= \frac{\hat{\boldsymbol{x}}}{s}\left[\frac{2}{1-\beta^2}\right], \qquad \text{so}$$

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t) = \frac{\lambda}{2\pi\epsilon_0}\frac{1-\beta^2}{1-\beta^2}\frac{\hat{\boldsymbol{x}}}{s} \qquad \text{or}$$

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t) = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\boldsymbol{x}}}{s}}$$

just as for a line charge at rest!

(b) Find the *magnetic* field of this string, using Eq. (10.69):

$$\vec{\boldsymbol{B}} = \frac{1}{c}\left(\vec{\boldsymbol{\mathcal{R}}}\times\vec{\boldsymbol{E}}\right) = \frac{1}{c^2}\left(\vec{\boldsymbol{v}}\times\vec{\boldsymbol{E}}\right)$$

where $\vec{\boldsymbol{\mathcal{R}}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'$. **ANSWER**: Well, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}$, so this is trivial:[1]

$$\boxed{\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0 I}{2\pi}\frac{\hat{\boldsymbol{y}}}{s}}$$

where $I = \lambda v$. (Again, the same result as for a steady current in magnetostatics.)

[1] Strictly speaking, Eq. (10.69) is for a point charge, and so should be applied separately to each charge element λdz. However, since $\vec{\boldsymbol{v}}$ has the same magnitude and direction for each, v comes outside the integral and all the non-y components of the individual cross products cancel out the same way the horizontal components of $\vec{\boldsymbol{E}}$ do.