ANTENNAS

http://musr.physics.ubc.ca/p401/pdf/AntennaPlus.pdf

by

Jess H. Brewer

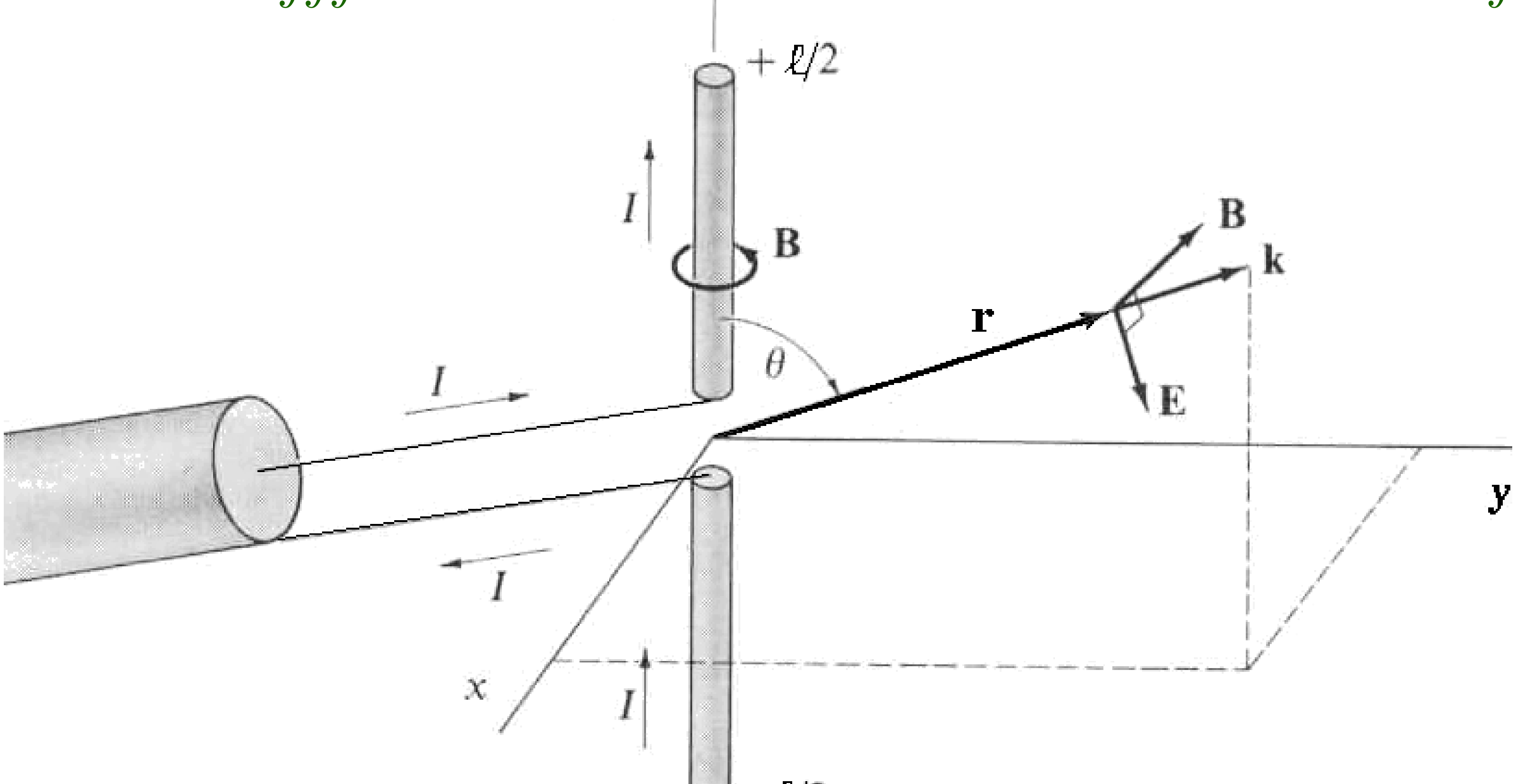
OUTLINE

Centre-Driven Linear Antenna

- Standing Wave of Current
- Side View of $\vec{z} \times \vec{r}$ Plane
- Half-Wave Antenna
- The Radiation Fields
- Radiated Power

Centre-Driven Linear Antenna

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}',t_r)}{\mathcal{R}} d\tau' \quad \text{here} \longrightarrow \quad \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0}{4\pi}\hat{\boldsymbol{z}}\int \frac{I(z',t_r)}{\mathcal{R}} dz'$$



$$\frac{1}{\mathcal{R}} \approx \frac{1}{r} \quad \text{and} \quad \mathcal{R} = t - \frac{\mathcal{R}}{c} \approx t - \frac{r}{c} + \frac{z'}{c}\,cos\theta$$

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Standing Wave of Current

Unlike the Electric Dipole antenna, where the charge flows onto spheres at the ends which can hold significant $\pm Q$, here we have simple wires with negligible capacitance, so *there is nowhere for the charge to go*. Moreover, we run the antenna at such a high ω that the instinct we cultivated for DC Circuits ($i.e.$ that I is the same throughout a circuit) is blatantly incorrect. By the time the ends of the wire "know" there is an increasing current, the driving voltage at the centre is already "trying" to reduce it.

This results in a "current wave" in the antenna with the same wavelength as the EM radiation it will generate, namely $\lambda = 2\pi c/\omega$. This wave is reflected from the ends, where of necessity $I = 0$ at all times ($i.e.$ the ends are *nodes*).

It is easiest to drive the resulting *standing wave* if the total length ℓ of the antenna is some multiple of half a wavelength:

$$\ell = m\,\frac{\lambda}{2} \qquad \text{so that} \qquad k \equiv \frac{\omega}{c} = m\,\frac{\pi}{\ell}\,. \tag{1}$$

That is, the antenna is driven *in resonance*.

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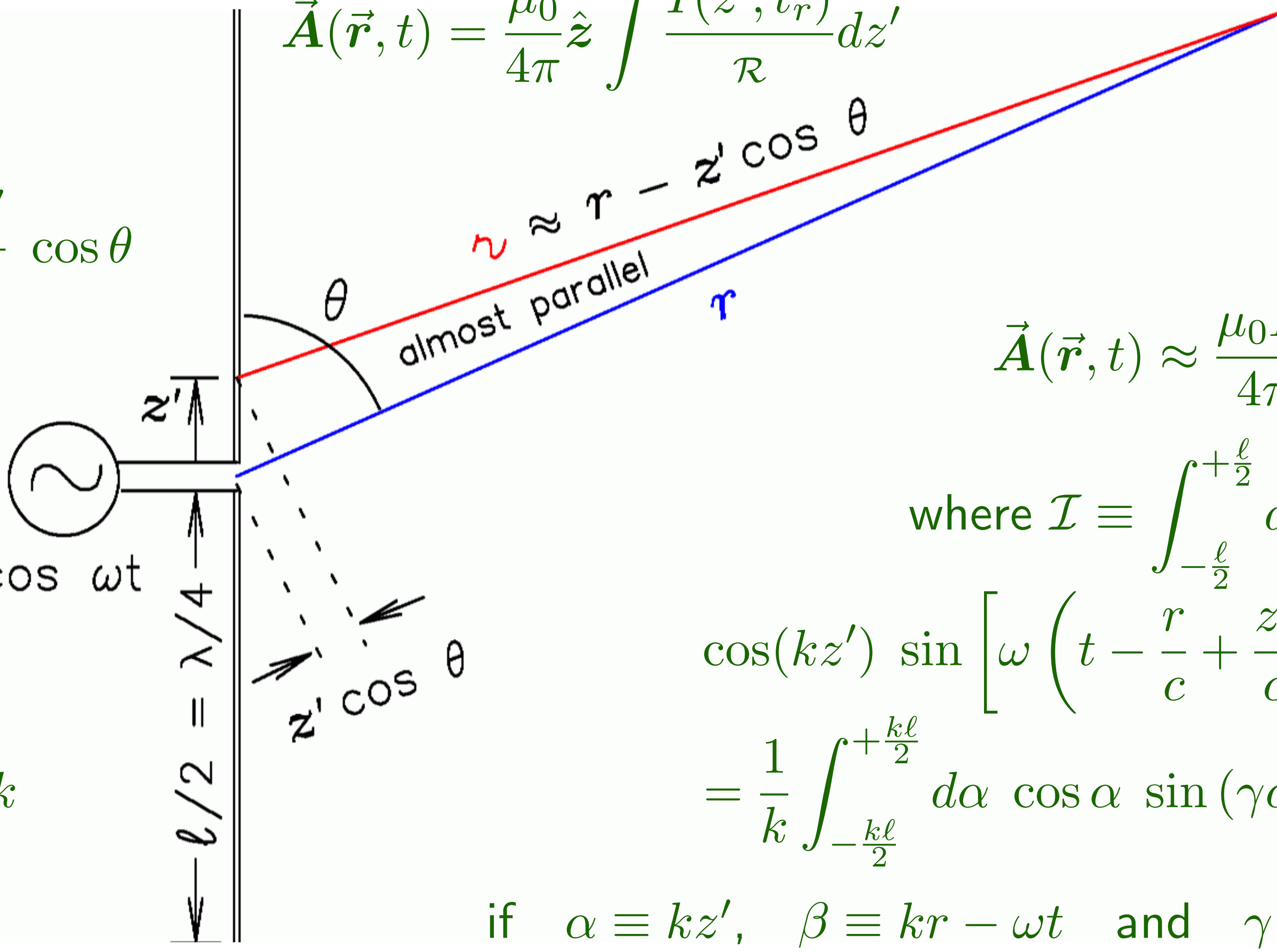
Side View of $\vec{z} \times \vec{r}$ Plane

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\hat{z}\int \frac{I(z',t_r)}{\mathcal{R}}dz'$$

$$t_r = t - \frac{\mathcal{R}}{c}$$
$$\approx t - \frac{r}{c} + \frac{z'}{c}\cos\theta$$



$$\vec{A}(\vec{r},t) \approx \frac{\mu_0 I_0}{4\pi}\frac{\hat{z}}{r}\mathcal{I}$$

where $\mathcal{I} \equiv \int_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} dz'$

$$\cos(kz')\ \sin\left[\omega\left(t - \frac{r}{c} + \frac{z'}{c}\cos\theta\right)\right]$$

$$= \frac{1}{k}\int_{-\frac{k\ell}{2}}^{+\frac{k\ell}{2}} d\alpha\ \cos\alpha\ \sin(\gamma\alpha - \beta)$$

if $\alpha \equiv kz'$, $\beta \equiv kr - \omega t$ and $\gamma \equiv \cos\theta$.

Note: $\frac{\omega}{c} \equiv k$

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$$\text{We have} \qquad \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) \approx \frac{\mu_0 I_0}{4\pi} \, \frac{\hat{\boldsymbol{z}}}{r} \, \mathcal{I} \tag{2}$$

$$\text{where} \qquad \mathcal{I} \equiv \frac{1}{k} \int_{-\frac{k\ell}{2}}^{+\frac{k\ell}{2}} d\alpha \;\, \cos\alpha \;\, \sin\left(\gamma\alpha - \beta\right) \tag{3}$$

$$\text{with} \qquad \alpha \equiv kz' \, , \qquad \beta \equiv kr - \omega t \qquad \text{and} \qquad \gamma \equiv \cos\theta \, . \tag{4}$$

$$\text{Expanding} \qquad \sin\left(\gamma\alpha - \beta\right) \; = \; \sin(\gamma\alpha) \, \cos\beta \; - \; \cos(\gamma\alpha) \, \sin\beta \tag{5}$$

and noting that neither $\cos\beta$ nor $\sin\beta$ depends on α (and so can be taken outside the integral), we can eliminate the first term because the integral of an odd function over a symmetric region about zero always vanishes. This leave the integral

$$\mathcal{I} = -\frac{\sin(kr - \omega t)}{k} \cdot 2 \int_0^{+\frac{k\ell}{2}} d\alpha \;\, \cos\alpha \;\, \cos(\gamma\alpha) \tag{6}$$

which we can look up ($e.g.$ at `http://integrals.wolfram.com/index.jsp`). . .

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Half-Wave Antenna

$$\ldots \text{ to get } \quad \mathcal{I} = -\frac{\sin(kr - \omega t)}{k} \cdot 2 \left[\frac{\gamma \cos\alpha \, \sin(\gamma\alpha) - \cos(\gamma\alpha) \, \sin\alpha}{\gamma^2 - 1} \right]_{\alpha=0}^{\alpha=\frac{k\ell}{2}} . \quad (7)$$

Now we impose the resonant condition (1): $\ell = m \dfrac{\lambda}{2}$ so that $k \equiv \dfrac{\omega}{c} = m \dfrac{\pi}{\ell}$ or $\dfrac{k\ell}{2} = \dfrac{m\pi}{2}$. The simplest case is the **Half-Wave Antenna ($m = 1$)** which gives $\cos\left(\dfrac{k\ell}{2}\right) = \cos(\pi/2) = 0$ and $\sin\left(\dfrac{k\ell}{2}\right) = \sin(\pi/2) = 1$ so that the quantity in square brackets in Eq. (7) is equal to $-\left[\dfrac{\cos(\gamma\alpha)}{\gamma^2 - 1}\right]$; since $\gamma \equiv \cos\theta$, $\gamma^2 - 1 = \sin^2\theta$ and (2) becomes

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) \approx -\frac{\mu_0 I_0 \ell}{2\pi^2} \left(\frac{\hat{\boldsymbol{z}}}{r}\right) \left[\frac{\cos\left(\frac{\pi}{2} \cos\theta\right)}{\sin^2\theta}\right] \sin(kr - \omega t) \, . \quad (8)$$

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The Radiation Fields

Taking the **curl** of Eq. (8) we get the magnetic field

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},t) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} \approx -\frac{\mu_0 I_0 \ell}{2\pi^2}\, \vec{\boldsymbol{\nabla}} \times \left\{ \left(\frac{\hat{\boldsymbol{z}}}{r} \right) \left[\frac{\cos\left(\frac{\pi}{2}\,\cos\theta\right)}{\sin^2\theta} \right] \sin(kr - \omega t) \right\} . \quad (9)$$

To evaluate this we express the oscillatory term as $\sin(kr - \omega t) = -i\, e^{i(kr-\omega t)}$ and refer to an argument developed in detail by Marion or Jackson that basically boils down to this: in the *far field* region, $e^{i(kr-\omega t)}$ changes so much more rapidly with position than the amplitude factors to its left that we may treat the outgoing wave as a *plane wave*, for which we know $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$ can be replaced by $ik\hat{\boldsymbol{k}} \times \vec{\boldsymbol{A}}$, where in this case $\vec{\boldsymbol{k}} = \hat{\boldsymbol{r}}\,\pi/\ell$. Since $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{z}} = -\hat{\boldsymbol{\phi}}\,\sin\theta$, we have

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},t) \approx \frac{\mu_0 I_0}{2\pi} \left(\frac{\hat{\boldsymbol{\phi}}}{r} \right) \left[\frac{\cos\left(\frac{\pi}{2}\,\cos\theta\right)}{\sin\theta} \right] e^{i(kr-\omega t)} . \quad (10)$$

and, as for any plane wave in free space, $\vec{\boldsymbol{E}} = c\vec{\boldsymbol{B}} \times \hat{\boldsymbol{k}} \; = \; c\,B\,\hat{\boldsymbol{\theta}}$.

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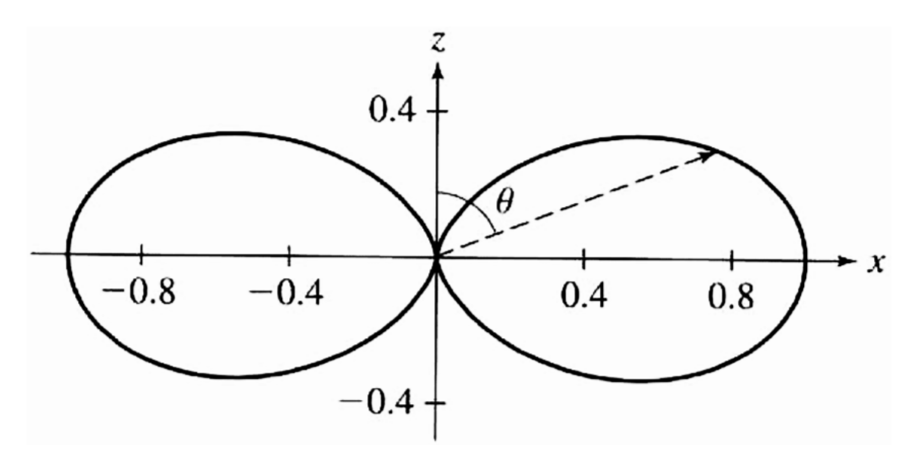
Radiated Power

For the half-wave antenna, Eq. (10) and the corresponding $\vec{\boldsymbol{E}}$ field give

$$\langle \vec{\boldsymbol{S}} \rangle = \langle \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} / \mu_0 \rangle \approx \frac{\mu_0 I_0^2 c}{8\pi^2} \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right]^2 , \tag{11}$$

which has the angular distribution depicted at right. If we integrate†

$$\int_0^\pi \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right]^2 \sin\theta \, d\theta \ = \ 1.22$$



we get the **total radiated power**:

$$\boxed{\langle P \rangle \approx \ 1.22 \, \frac{\mu_0 I_0^2 c}{4\pi}} . \tag{12}$$

† You're right, this is not an easy integral! But it *can* be found.

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There are, of course, different radiation patterns for other resonances. The **full-wave antenna** $(m = 2)$, for instance, produces a double-lobed angular distribution of the radiated power, with a node at $\theta = \pi/2$. Since most radio stations wish to broadcast more or less horizontally from a vertical tower, the half-wave antenna is a better choice.

More exotic angular patterns can be achieved using a **phased array** of $(e.g.)$ half-wave transmitters. You have seen one example in your First Year Optics course: a linear array of equally spaced radiators, all in phase: this is essentially the same as the (misnamed) *diffraction grating*.

Today cell phone companies use other geometrical arrangements of transmitters whose phase can be adjusted as desired to send narrow **beams** to several clients in the same cell, thereby "multiplexing" each available frequency and increasing the number of clients that can be served simultaneously.

Remember also that a **receiver** is just an "antenna in reverse". . . .

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