E PHYS 401 M Lecture # 2

- **Logistics & Leftovers:** *WebCT ?*
- **Review of Electrostatics & Magnetostatics, cont'd**
  - **Coulomb's (↔ Gauss') Law & Biot-Savart (↔ Ampère's) Law**
  - "TOOLKIT" for **Electrostatics**:
    - Electrostatics and **Conductors**
    - Equations of **Poisson** & **Laplace**
    - Method of **Images**
    - **Multipole** expansions
  - **... for Magnetostatics:**
    - The Vector Potential ***A***
    - **Multipole** expansions
    - . . .

Conductors & Electrostatics

$\boldsymbol{J} = \sigma \boldsymbol{E}$ until charge redistributes itself to cancel out $\boldsymbol{E}$. Consequently

$\boldsymbol{E} = 0$ inside a conductor. Since $\boldsymbol{E} = -\vec{\nabla} V$, $V = \text{const.}$ in a conductor.

By Gauss' law, if $\boldsymbol{E} = 0$ then $\rho = 0$ as well. All charges go to surfaces.

At any interface, $\boldsymbol{E}_{\text{above}} - \boldsymbol{E}_{\text{below}} = \sigma$ (σ = surface charge density)‡.

Just outside a conductor, $E_{\parallel} = 0$ & $E_{\perp} = \sigma$.

Thus the surfaces of conductors are key **boundaries** between regions where (usually) Laplace's equation $\nabla^2 V = 0$ applies. See next page.

Note: all the above assumes **electrostatics** (no steady currents applied)!

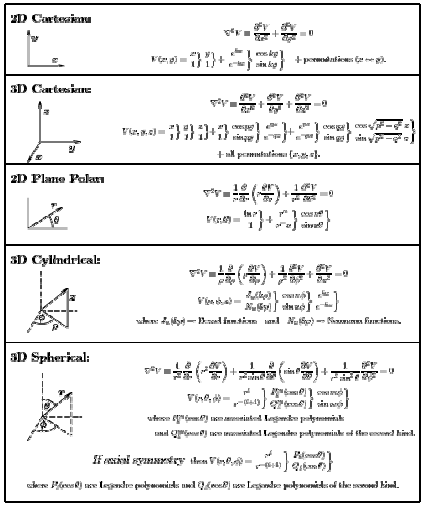
‡ Notational ambiguity: σ the surface charge density vs. σ the conductivity!

Solutions to Laplace's Equation: $\nabla^2 V = 0$

In general, $\nabla^2 V = \dfrac{\rho}{\epsilon_0}$ (Poisson's equation)

but most practical problems involve free space, dielectrics or conductors, where $\rho = 0$ in the regions of interest.

We solve the **differential equation** by **separation of variables** in an appropriate coordinate system, then **try** a linear combination of all the (finite variety of) solutions in that geometry, using the **equipotentials** of conducting surfaces as **boundary conditions**.



Method of Images

If a combination of the field due to real charges and that due to "pretend charges" **inside** the conductor would give a new field that is everywhere **perpendicular to the surface of the conductor** (making that surface an **equipotential**), then that is the field in the region **outside** the conductor. (Uniqueness theorem)

Not a useful "brute force" technique, but great for "**tricks**"!

Examples:

Point charge beside a conducting **plane**

Point charge beside a conducting **sphere**

The Vector Potential

Just as we set $\boldsymbol{E} = -\vec{\nabla} V$, we can express $\boldsymbol{B}$ as the **curl** of a **vector** potential $\boldsymbol{A}$: $\boldsymbol{B} = \vec{\nabla} \times \boldsymbol{A}$. Plugging this into Ampère's law yields $\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J} + \vec{\nabla} \times (\vec{\nabla} \cdot \boldsymbol{A})$. We can always choose $\vec{\nabla} \cdot \boldsymbol{A} = 0$ to make that last term go away, leaving $\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$, which (by analogy with Poisson's equation for V) has the general solution

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{\mathcal{R}} \, d\tau'$$

Multipole Expansions

When the test point at $\boldsymbol{r}$ is far away and the source region (the range of $\boldsymbol{r}'$) is tiny by comparison ($\boldsymbol{r}' \ll \boldsymbol{r}$) we can treat the source region as "the origin" and expand the potentials in powers of $(\boldsymbol{r}'/\boldsymbol{r})$, yielding for the scalar (electrostatic) potential

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{\boldsymbol{r}}')}{\mathcal{R}} \, d\tau' = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \iiint (r')^n \, P_n(\cos\theta') \, \rho(\vec{\boldsymbol{r}}') \, d\tau'$$

and for the vector (magnetostatic) potential

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{\mathcal{R}} \, d\tau' = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \iiint (r')^n \, P_n(\cos\theta') \, \vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}') \, d\tau'$$

where $P_n(\cos\theta')$ are Legendre polynomials.