# Experiment 1

# Electromagnetic Skin Depth of Metals

## 1.1 Introduction

It is common knowledge that electromagnetic radiation does not pass easily through a metal. The electric field associated with the wave generates relatively large currents in the metal which flow in such a way as to shield the metal interior from the radiation. It is the purpose of this experiment to examine the so called 'skin depth' problem quantitatively.

Metals typically do not strongly attenuate *AC magnetic fields* in the frequency range $0 \to 10^6$ Hz. In fact, this magnetic shielding effect does not become important for ordinary metals for frequencies less than $\sim 10^9$ Hz. The shielding effect that is typically discussed in connection with low frequency and radio frequency circuits is *electrostatic shielding*, which is due to surface charges which are induced on any closed metal object located in an electrostatic field.

## 1.2 The Skin Depth Problem

Let a metal cylinder be immersed in a uniform magnetic field which is varying with time as $e^{\imath\omega t}$. The magnetic field is parallel to the cylinder axis and this direction is taken to be the $z$ direction of a system of polar co-ordinates, as seen in Fig. 1.1. The cylinder is hollow and has an outer diameter $2R_2$ and an inner diameter $2R_1$. The problem is to calculate the amplitude and phase of the magnetic field everywhere inside the cylinder.

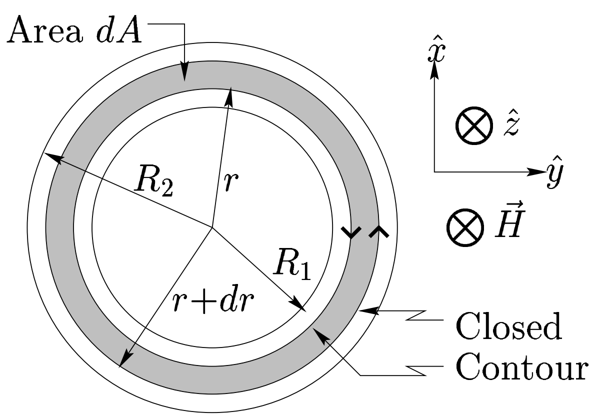


Figure 1.1: The geometry used in the cylindrical skin depth problem.

One always begins this sort of problem by invoking Maxwell's equations (in SI units):

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{1.1}$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \epsilon \frac{\partial \vec{E}}{\partial t} \tag{1.2}$$

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon \tag{1.3}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{1.4}$$

where it has been assumed that the relation between $\vec{B}$ and $\vec{H}$ can be written $\vec{B} = \mu\vec{H}$. The problem presented here has cylindrical symmetry so both $\vec{E}$ and $\vec{H}$ depend only upon the radius $r$. Moreover, $\vec{H}$ has only one component, $\vec{H} = H_z$. This component does not depend upon $z$, and so the condition $\nabla \cdot \vec{H} = 0$ is satisfied.

It is easy to apply Eqn. 1.1 to a ring of radius $r$ and thickness $dr$ (see Fig. 1.1) if the equation is written in integral form and one employs Stokes' theorem:

$$\oint_C \vec{E} \cdot \vec{ds} = -\mu \frac{d}{dt} \iint_A \vec{H} \cdot \vec{dS}$$

By appropriate integration over the complete contour $C$ and area $A$ one obtains the following result:

$$2\pi(r + dr)E_\theta(r + dr) - 2\pi r E_\theta(r) = -\mu \frac{d}{dt}\Big(2\pi r \cdot dr \cdot H_z(r)\Big)$$

$$2\pi \frac{d}{dr}\Big(rE_\theta\Big) = -\mu \frac{d}{dt}\Big(2\pi r H_z(r)\Big)$$

It will be assumed that all quantities vary with time like $e^{\imath\omega t}$. One can now use this explicit time dependence.

$$\frac{1}{r}\frac{d}{dr}(rE_\theta) = \imath\mu\omega H_z \tag{1.5}$$

Similarly, from Eqn. 1.2 and the rectangular closed contour shown in Fig. 1.2,

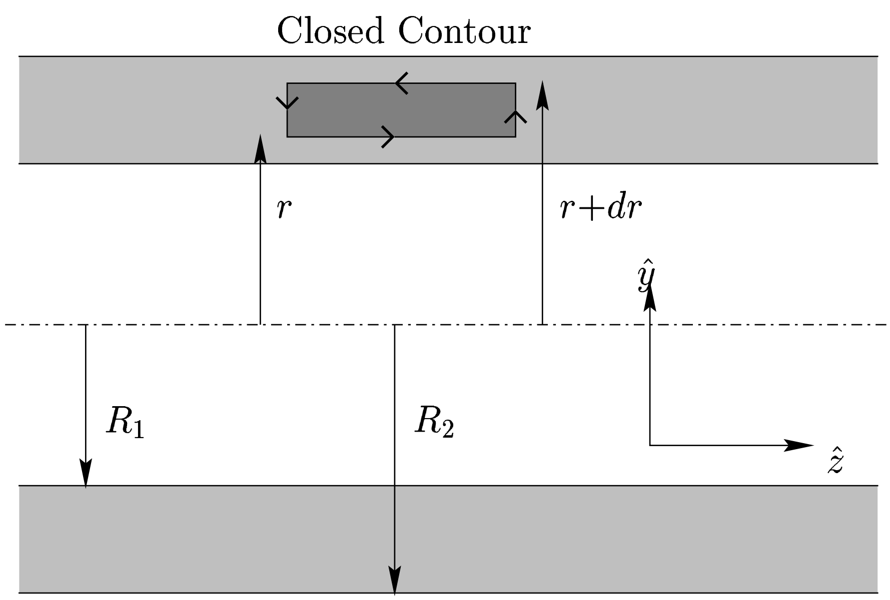


Figure 1.2: A rectangular closed contour.

$$\frac{dH_z}{dr} = -j_\theta + \imath\omega\epsilon E_\theta \tag{1.6}$$

Equations 1.5 and 1.6 can be combined to give an expression for $E_\theta$:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rE_\theta)\right] + \epsilon\mu\omega^2 E_\theta = -\imath\mu\omega j_\theta \tag{1.7}$$

For an isotropic metal, one can typically relate current and electric fields using Ohm's law in the form $\vec{j} = \sigma\vec{E}$, where $\sigma$ is the (assumed scalar) conductivity of the material. Upon substituting and rearranging, one obtains the following:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rE_\theta)\right] + \mu\omega(\epsilon\omega + \imath\sigma)E_\theta = 0 \tag{1.8}$$

At sufficiently low frequency, $\epsilon$ is related to the polarizability of the material, hence it is proportional to the amount of bound charge within the system. On the other hand, $\sigma$ is proportional to the amount of free charge. If free charge dominates the electromagnetic response (metallic behaviour), then one can assume that $\epsilon\omega \ll \sigma$ and ignore the bound charge response. In this regime, Eqn. 1.8 becomes

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rE_\theta)\right] + \imath\omega\mu\sigma E_\theta = 0 \tag{1.9}$$

which will have two distinct types of solutions outside ($\sigma = 0$) and inside ($\sigma \neq 0$) the metal.

• Vacuum ($\sigma = 0$)

Eqn. 1.9 will have the following simple solution:

$$E_\theta(r) = \frac{A_v r}{2} + \frac{B_v}{r} \tag{1.10}$$

where $A_v$ and $B_v$ are constants to be determined by boundary conditions. One can now solve for the magnetic field outside of the metal by using Eqn. 1.5.

$$H_{z(vacuum)}(r) = \frac{A_v}{\imath\mu\omega} = constant \tag{1.11}$$

Therefore, if a magnetic field $\vec{H} = H_o\hat{z}$ exists outside of the cylinder, then the field inside the hollow of the cylinder will have a constant amplitude given by Eqn. 1.11. Note that $A_v$ can be a complex quantity, thus allowing for both attenuation of the amplitude and a phase shift relative to $\vec{H}$ outside of the cylinder.

• Metal ($\sigma \neq 0$)

For this case, Eqn. 1.9 is a particular form of Bessel's equation. In order to place Eqn. 1.9 into a more standard form, it shall be necessary to make a change of variables. Let

$$v = r\sqrt{\imath\omega\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}}r(\imath + 1) = k_o r(\imath + 1)$$

Substituting into Eqn. 1.9 then gives

$$\frac{d^2E_\theta}{dv^2} + \frac{1}{v}\frac{dE_\theta}{dv} + \left(1 - \frac{1}{v^2}\right)E_\theta = 0 \tag{1.12}$$

which is Bessel's equation of order 1 whose solutions are designated $J_1(z)$ and $Y_1(z)$. The general solution is then

$$E_\theta = A_m J_1(v) + B_m Y_1(v) \tag{1.13}$$

where $A_m$ and $B_m$ are constants to be determined by the boundary conditions. The magnetic field inside of the metal can be obtained from Eqn. 1.5 and the recurrence relation for derivatives of Bessel functions:

$$\frac{dJ_m}{dz} = J_{m-1}(z) - \frac{m}{z}J_m(z)$$

from which it follows that

$$H_{z(metal)} = \frac{k_o(\imath + 1)}{\imath\mu\omega}\Big(A_m J_0(v) + B_m Y_0(v)\Big) \tag{1.14}$$

Now that expressions for the magnetic field both inside and outside of the metal have been obtained, one can return to the original task - to determine the attenuation and phase shift of the field $H_i$ inside of the cylinder with respect to the applied field $H_o$ outside of the cylinder. Given that $H_z$ must be continuous across the vacuum-metal interfaces, one can determine expressions for the constant fields $H_i$ and $H_o$ by evaluating Eqn. 1.14 at radii $R_1$ and $R_2$, respectively. Let $v_1 = k_o R_1(\imath + 1)$ and $v_2 = k_o R_2(\imath + 1)$ and then determine the ratio $\alpha = H_i/H_o$:

$$\alpha = \frac{H_i}{H_o} = \frac{J_0(v_1) + (B_m/A_m)Y_0(v_1)}{J_0(v_2) + (B_m/A_m)Y_0(v_2)} \tag{1.15}$$

Finally, the ratio $B_m/A_m$ can be rewritten by invoking the integral form of Eqn. 1.1 and using a circle of radius $R_1$ in the $(r, \theta)$ plane.

$$\oint_C \vec{E} \cdot d\vec{s} = -\mu \frac{d}{dt} \iint_A \vec{H} \cdot d\vec{S}$$

$$2\pi R_1 E_\theta(R_1) = -\mu \frac{d}{dt}\left(\pi R_1^2 H_i\right)$$

Upon substituting $E_\theta$ from Eqn. 1.13 and $H_i$ from Eqn. 1.14, both evaluated at radius $R_1$, one can solve for $B_m/A_m$.

$$B_m/A_m = -\frac{(k_o R_1/2)(\imath + 1)J_0(v_1) + J_1(v_1)}{Y_1(v_1) + (k_o R_1/2)(\imath + 1)Y_0(v_1)} \tag{1.16}$$

Substituting Eqn. 1.16 into Eqn. 1.15 then yields a complete expression for the complex attenuation coefficient $\alpha$.

$$\alpha = \frac{J_0(v_1)Y_1(v_1) - J_1(v_1)Y_0(v_1)}{\left(J_0(v_2)Y_1(v_1) - J_1(v_1)Y_0(v_2)\right) + (k_o R_1/2)(\imath + 1)\left(J_0(v_2)Y_0(v_1) - J_0(v_1)Y_0(v_2)\right)} \tag{1.17}$$

In principle, the skin depth problem for a hollow circular cylinder has been solved. Given the dimensions of the pipe and the conductivity of the metal, the quantities $k_o$, $v_1$ and $v_2$ are known. With modern software it is a relatively simple matter to use Eqn. 1.17 in its present form, however it shall be instructive to take a high frequency limit using limiting forms for the Bessel functions. In the limits $|z| \to \infty$ and $|Arg[z]| < \pi$,

$$J_m(z) \to \sqrt{2/\pi z}\cos(z - m\pi/2 - \pi/4)$$

$$Y_m(z) \to \sqrt{2/\pi z}\sin(z - m\pi/2 - \pi/4)$$

In this asymptotic limit, which corresponds to $k_o R_1, k_o R_2 \to \infty$, the attenuation coefficient becomes

$$\alpha \to 2\sqrt{\frac{R_2}{R_1}} \frac{e^{(\imath - 1)k_o(R_2 - R_1)}}{1 + (1 - \imath)k_o R_1/2} \tag{1.18}$$

which can be rewritten in terms of a polar quantity $\alpha \equiv \rho e^{\imath\phi}$ as follows:

$$\rho = 2\sqrt{\frac{R_2}{R_1}} \frac{e^{-k_o(R_2 - R_1)}}{\sqrt{1 + k_o R_1 + k_o^2 R_1^2/2}} \tag{1.19}$$

$$\phi = k_o(R_2 - R_1) + \arctan\left(\frac{k_o R_1}{2 + k_o R_1}\right) \tag{1.20}$$

Eqns. 1.19 and 1.20 are accurate to $\sim 1\%$ for $k_o R_1 > 5$.

Thus an explicit theoretical expression for the complex attenuation coefficient for a cylindrical geometry has been constructed. However, Eqns. 1.19 and 1.20 do not yield a simple physical interpretation. For this purpose, the student is asked to solve a simpler problem in which a magnetic field $\vec{H} = H_o\hat{z}$ is oriented parallel on one side of an infinite plane metal slab of thickness $d$, as shown in Fig. 1.3.

The student should be able to demonstrate that the magnetic field on the opposite side of the slab is

$$\vec{H} = H_o e^{(\imath - 1)k_o d}\hat{z}$$

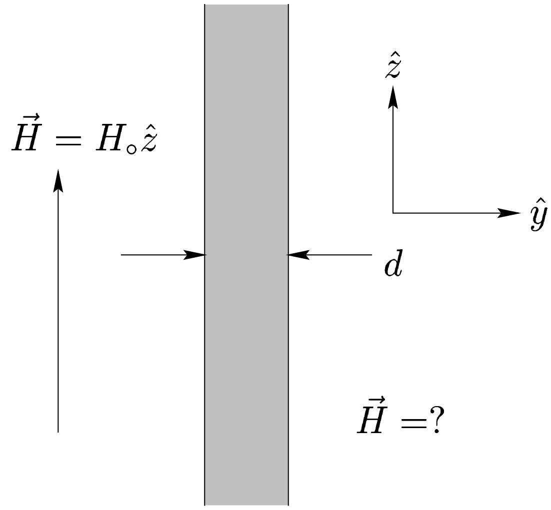


Figure 1.3: The skin depth problem for an infinite slab of metal.

and then one can easily recognize that $\delta_o \equiv 1/k_o$ is the lengthscale over which a magnetic field is extinguished by a factor of $e^{-1}$ by a metallic conductor. The lengthscale $\delta_o$ is known as the *skin depth* and is intimately linked to the conductivity of a metal.

$$\delta_o = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{1.21}$$

Though the infinite slab problem provides a more direct means of interpreting the relation between a complex attenuation coefficient in terms of the skin depth, there are practical difficulties which make the cylindrical geometry easier to implement. The student is asked to consider those practical difficulties that would be encountered if one attempted to implement an infinite slab version of the skin depth experiment.

## 1.3 Experiment

A coil approximately 6" long, 2" in diameter and wound with approximately 10 turns/cm is connected in series with a 50 Ω resistor to a *HP 3324A* synthesizer. This system is used to generate alternating magnetic fields over the frequency range 100 Hz to 40 kHz. It is worthwhile to roughly calculate the frequency response of this LR circuit. If one includes stray capacitance, what will happen to the frequency response of the circuit?

A pickup coil of approximately 200 turns of #38 *Formex* insulated copper wire has been wound on a 3/8" diameter nylon rod to probe the field in the primary coil. The small EMF generated by the pickup coil is amplified and then displayed on a *Textronix 2232* digital storage scope. A LabView-based program, `C:\Physics 352\Skin Depth.exe`, is used to input the data from the scope to the PC via an *IEEE-488* bus. The program will guide you through the data collection process.

The object of the experiment is to measure the phase and amplitude of the alternating magnetic field inside a metal pipe relative to the field outside of the pipe. Thus you will need to perform measurements both with and without the metal pipe present. The data are to be compared with the high frequency approximation for $\alpha$ via Eqns. 1.19 and 1.20.

To begin, use either an aluminum or copper pipe. Locate reasonable conductivity values in a reliable reference and then fit the data to the model to determine an experimental measure of $\sigma$. How well does the skin depth model work?

Now repeat the experiment using a steel pipe. Note that the magnetic shielding has improved considerably, however the skin depth model is incapable of fitting the data very well. Why does steel behave so differently than the metals studied previously?

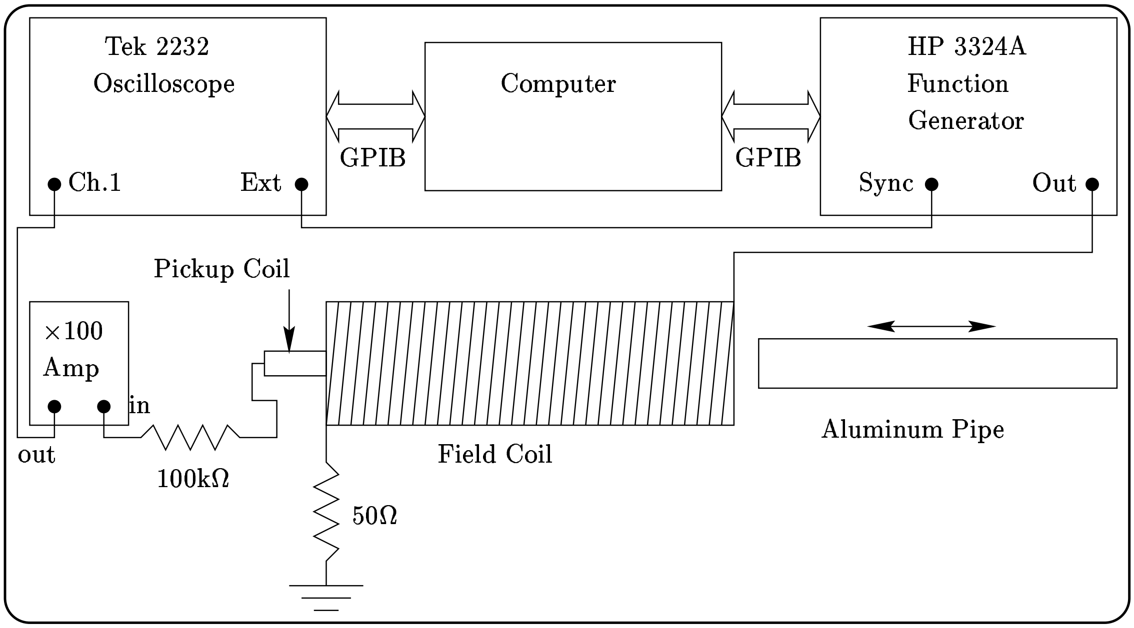


Figure 1.4: A block diagram of the skin depth apparatus.

## 1.4 Selected References

Griffiths, David J. *Introduction to Electrodynamics.* Prentice Hall Press, Englewood Cliffs USA, 1989.

Weast, Robert C. (ed.) *CRC Handbook of Chemistry and Physics.* CRC Press, Boca Raton USA.