COMPLEX EXPONENTIALS

In your first exposure to SIMPLE HARMONIC MOTION and WAVES you probably saw only the *real* sinusoidal functions $\sin\theta$ and $\cos\theta$ (where $\theta = kx - \omega t + \phi$, the *phase* of an oscillation). This was reasonable enough, since all the phenomena of classical mechanics are in fact real, at least in the mathematical sense. Whether they are real in the colloquial sense is subject to discussion....

In QUANTUM MECHANICS, which we claim describes the way the *real* world *real*ly works, things are *not* always real in the mathematical sense. Well, "things" are always real, if by "things" you mean physical observables, but the things you have to talk about to make *predictions* about the real "things" — or at least about what you are likely to measure if you observe one — those things are *not* real; they are almost always *complex*. Sort of like that sentence, eh? No, *mathematically complex*. That is, complex in the mathematical sense, *i.e.* having a real part and an imaginary part.

With that introduction to QUANTUM MECHANICS I should have produced the proper state of confusion one needs to approach the subject. But for now I would like to demonstrate a few simple properties of the most remarkable function ever invented: the exponential function, $\exp(x) \equiv e^x$.

$e^{i\theta}$.1 Sines, Cosines, Exponentials

It is helpful to remember the definition of the exponential function in terms of a power series:

$$\exp(x) \equiv e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$

If we let x be *imaginary*, $x = i\theta$ (where θ is real), then this can be written

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{6} + \frac{\theta^4}{24} + i\frac{\theta^5}{120} - \dots$$

which ought to remind you (doesn't it?) of the series expansions for the sinusoidal functions:

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \dots$$
&
$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots$$

Note that these expansions perform a sort of "leapfrog" between even and odd terms. Putting them all together we can easily see that

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

and, since $\cos(-\theta) = \cos\theta$ while $\sin(-\theta) = -\sin\theta$,

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

These simple equivalences are, to my mind, among the most astonishing relationships in all of mathematics. Why? Because they show an intimate relationship between two functions which would seem at first glance to have absolutely nothing in common: the monotonically increasing or decreasing exponential function $e^{\pm x}$ and the sinusoidally oscillating sin and cos functions!

We can also invert the relationship and obtain a definition for the sin and cos functions in terms of exponentials:

$$\boxed{\cos\theta \equiv \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

$$\boxed{\sin\theta \equiv \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

Let's review the most important (for Physics, anyway) property of the exponential function: it is its own derivative! $\frac{d}{dx}e^x = e^x$. If $x = kt$

then we "pull out an extra factor of k" with each derivative with respect to t:

$$\boxed{\frac{d^n}{dt^n}\, e^{kt} \;=\; k^n\, e^{kt}}$$

This latter property (which, by the way, works just as well for complex k as for real k) establishes the connection between the *derivatives* of $e^{i\omega t}$ (for instance) and those of $\sin(\omega t)$ and $\cos(\omega t)$:

$$\begin{aligned} \frac{d^2}{dt^2}\, e^{i\omega t} &= -\omega^2\, e^{i\omega t} \\ \frac{d^2}{dt^2}\, \cos(\omega t) &= -\omega^2\, \cos(\omega t) \\ \frac{d^2}{dt^2}\, \sin(\omega t) &= -\omega^2\, \sin(\omega t) \end{aligned}$$

You will recall how useful these second-derivative properties were in SIMPLE HARMONIC MOTION and WAVES. Well, you ain't seen nothing yet!

$e^{i\theta}$.2 Complex Angles

What happens if we take the exponential of a quantity that is neither pure real nor pure imaginary, but a little of both? We can do this several ways, but in view of our interest in waves I will put it this way: suppose that instead of θ we have an argument $z = (\omega + i\lambda)t$ where ω, λ and t are all real. Then

$$e^{iz} \;=\; e^{i(\omega+i\lambda)t} \;=\; e^{i\omega t} \cdot e^{-\lambda t}$$

That is, we have an oscillatory function multiplied by an exponentially decaying "envelope" function — the phenomenon of DAMPED OSCILLATIONS that describes virtually every actual case of oscillatory motion.

$e^{i\theta}$.3 Hyperbolic Functions

Another question arises if we are familiar with the HYPERBOLIC FUNCTIONS

$$\boxed{\cosh x \;\equiv\; \frac{1}{2}\left(e^{x} \;+\; e^{-x}\right)}$$

$$\boxed{\sinh x \;\equiv\; \frac{1}{2}\left(e^{x} \;-\; e^{-x}\right)}$$

These are so similar to the definitions of the sin and cos in terms of complex exponentials that we suspect a connection between cosh and cos that is deeper than just the fact that the names are so similar (which should of course have made us suspicious in the first place). I will leave it as a (trivial) exercise for the reader to show that

$$\cos\theta = \cosh(i\theta) \qquad\qquad i\sin\theta = \sinh(i\theta)$$

$$\cosh x = \cos(ix) \qquad\qquad i\sinh x = \sin(ix).$$