# Equations of Motion

In the previous chapter we explored the process of *emergence* of new paradigms in Mechanics, using various mathematical identities to transform Newton's SECOND LAW into new equations whose left- and right-hand sides were given names of their own, like *impulse, momentum, work, energy, torque* and *angular momentum*. Eighteenth-Century physicists then learned to manipulate these "new" concepts in ways that greatly clarified the behaviour of objects in the material universe. As a result, previously mysterious or counterintuitive phenomena began to make sense in terms of *simple, easy-to-use models*, rather than long involved calculations. This is the essence of what Physics is all about. We work hard to make todays's difficult tasks easier, so that we will have more free time and energy tomorrow to work hard to make tomorrow's difficult tasks easier, so that....

Meanwhile, these new words made their way into day-to-day language and introduced new paradigms into society, whose evolution in "The Age of Reason" might have followed other paths were it not for Newton's work.[1] The effects of a more versatile and effective science of Mechanics were also felt in blunt practical terms: combined with the new science of Thermodynamics (to be discussed in a later chapter), Mechanics made possible an unprecedented growth of Mankind's ability to push Nature around by brute force, a profitable enterprise (in the short term) that led to the Industrial Revolution. Suddenly people no longer had to accept what Nature dealt, which enhanced their health and wealth considerably — but in taking new cards they found they also had a new dealer who was more merciless than Nature had ever been: Greed.

Here arises a perennial question: are the evils of "technology abuse," from pollution to exploitation to weapons of war, the "fault" of scientists who create the conceptual tools that make technology possible?[2] My own opinion is that we scientists have a responsibility for our creations in much the same way that parents have a responsibility for their children: we try to provide a wholesome and enlightened atmosphere in which they can grow and fulfill all their potential, offering our guidance and advice whenever it will be accepted, and setting the best example we can; but in the end ideas are like people — they will determine their own destiny. The best scientists can do to guide the impact of their ideas on society is to make sure the individual members of society have the opportunity to learn about those ideas. Whether anyone takes advantage of that opportunity or not is out of our control. Whether irresponsible or malign individuals make evil use of our ideas is also out of our control, though we can do our best to dissuade them.[3]

## 12.1 "Solving" the Motion

Getting back to the subject of Mechanics...

One of the reasons the paradigms in the previous chapter emerged was that physicists were always trying to "solve" certain types of "problems" using Newton's SECOND LAW,[4]

$$F = m\ddot{x}$$

[1] Then again, maybe subtle sociological evolution had already made these changes inevitable and Newton was just the vehicle through which the emergent paradigms of the day infiltrated the world of science. Let's do the Seventeenth and Eighteenth Centuries over again without Newton and see how it comes out!

[2] I presume that I do not need to point out the distinction between *Science* and *Technology*. Even though politicians seem to be fond of the word "scienceandtechnology," I feel sure my readers are intelligent enough to find such a juxtaposition humourous.

[3] Some people feel that we should be prevented from *having* new ideas until those ideas have been "cleared" as innocuous. This would be hilarious if it weren't so dangerous.

[4] Let's limit our attention to *one dimensional* problems for the duration of this chapter, to keep things simple and avoid the necessity of using vector notation.

This equation can be written

$$\ddot{x} = \frac{1}{m}\, F \tag{1}$$

to emphasize that it described a relationship between the *acceleration* $\ddot{x}$, the inertial coefficient $m$ [usually constant] and the force $F$. It is conventional to call an equation in this form the "**equation of motion**" governing the problem at hand. When $F$ is *constant* [as for "local" gravity] the "solution" to the equation of motion is the well-known set of equations governing *constant acceleration*, covered in the chapter on FALLING BODIES. Things are not always that simple, though.

Sometimes the problem is posed in such a way that the force $F$ is explicitly a function of *time*, $F(t)$. This is not hard to work with, at least in principle, since the equation of motion (1) is then in the form

$$\ddot{x} = \frac{1}{m}\, F(t) \tag{2}$$

which can be straightforwardly *integrated* [assuming one knows a function whose time derivative is $F(t)$] using the formal operation

$$v(t) \;\equiv\; \dot{x} \;\equiv\; \int_0^t \ddot{x}\, dt \;=\; \frac{1}{m} \int_0^t F(t)\, dt \tag{3}$$

— which, when multiplied on both sides by $m$, leads to the paradigm of *Impulse and Momentum*.

In other cases the problem may be posed in such a way that the force $F$ is explicitly a function of *position*, $F(x)$. Then the equation of motion has the form

$$\ddot{x} = \frac{1}{m}\, F(x) \tag{4}$$

which can be converted without too much trouble [using the identity $a\, dx = v\, dv$ ] into the paradigm of *Work and Energy*.

### 12.1.1 Timing is Everything!

If the equation of motion is the "question," what constitutes an "answer"? Surely the most *convenient* thing to know about any given problem is the *explicit time dependence of the position*, $x(t)$, because if we want the *velocity* $v(t) \equiv \dot{x}$, all we have to do is take the first time derivative — which may not be entirely trivial but is usually much easier than integrating! And if we want the *acceleration* $a(t) \equiv \dot{v} \equiv \ddot{x}$, all we have to do is take the time derivative again. Once you have found the acceleration, of course, you also know the net *force* on the object, by NEWTON'S SECOND LAW. A problem of this sort is therefore considered "solved" when we have discovered the explicit function $x(t)$ that "satisfies" the equation of motion.

For example, suppose we know that

$$x(t) = x_0 \cos(\omega t), \tag{5}$$

where $\omega$ is some constant with units of *radians/unit time*, so that $\omega t$ is an *angle*. The time derivative of this is the velocity

$$\dot{x} \equiv v(t) = -\omega\, x_0 \sin(\omega t)$$

[look it up if needed] and the time derivative of *that* is the acceleration

$$\ddot{x} \equiv \dot{v} \equiv a(t) = -\omega^2\, x_0 \cos(\omega t).$$

Note that the right-hand side of the last equation is just $-\omega^2$ times our original formula for $x(t)$, so we can also write

$$\ddot{x} = -\omega^2\, x. \tag{6}$$

Multiplying through both sides by the mass $m$ of the object in motion gives

$$ma = F = -m\omega^2\, x,$$

which ought to look familiar to you: it is just HOOKE'S LAW with a force constant $k = m\omega^2$. Rearranging this a little gives

$$\omega = \sqrt{k/m},$$

which may also look familiar.... More on this later. Note, however, that we can very easily deduce what is going on in this situation, including the type of force being applied, just from knowing $x(t)$. That's why we think of it as "the solution."

### 12.1.2 Canonical Variables

Let's write the equation of motion in a *generalized* form,

$$\ddot{q} = \frac{1}{m} F \tag{7}$$

where I have used "$q$" as the "canonical coordinate" whose second derivative ($\ddot{q}$) is the "canonical acceleration." Normally $q$ will be the spatial position $x$ [measured in units of length like metres or feet], but you have already seen one case (rotational kinematics) in which "$q$" is the *angle* $\theta$ [measured in radians], "$m$" is the *moment of inertia* $I_O$ and "F" is the *torque* $\Gamma_O$; then a completely analogous set of equations pertains. This turns out to be a quite common situation. Can we describe simply how to go about formulating the equations of motion for "systems" that might even be completely different from the standard objects of Classical Mechanics?

In general there can be any number of canonical coordinates $q_i$ in a given "system" whose behaviour we want to describe. As long as we have an explicit formula for the *potential energy* $V$ in terms of one or more $q_i$, we can define the *generalized force*

$$Q_i = -\frac{\partial V}{\partial q_i} \tag{8}$$

If we then generalize the "inertial coefficient" $m \to \mu$, we can write out $i^{\rm th}$ equation of motion in the form

$$\ddot{q}_i = \frac{Q_i}{\mu} \tag{9}$$

which in most cases will produce a valid and workable solution. There is an even more general and elegant formulation of the canonical equations of motion which we will discuss toward the end of this chapter.

I am not really sure how the term *canonical* came to be fashionable for referring to this abstraction/generalization, but Physicists are all so fond of it by now that you are apt to hear them using it in all their conversations to mean something like *archetypal*: "It was the canonical Government coverup..." or "This is a canonical cocktail party conversation...."

### 12.1.3 Differential Equations

What we are doing when we "solve the equation of motion" is looking for a "solution" [in the sense defined above] to the *differential equation* defined by Eq. (7). You may have heard horror stories about the difficulty of "solving differential equations," but it's really no big deal; like long division, basically you can only use a trial-and-error method: does this function have the right derivative? No? How about this one? And so on. Obviously, you can quickly learn to recognize certain functions by their derivatives; more complicated ones are harder, and it doesn't take much to stump even a seasoned veteran. The point of all this is that "solving differential equations" is a difficult and arcane art only if you want to be able to solve *any* differential equation; solving the few *simple* ones that occur over and over in physics is no more tedious than remembering multiplication tables. Some of the other commonly-occuring examples have already been mentioned.

### 12.1.4 Exponential Functions

You have seen the procedure by which a new function, the *exponential* function $q(t) = q_0 \exp(kt)$, was constructed from a power series just to provide a solution to the differential equation $\dot{q} = k\, q$. (There are, of course, other ways of "inventing" this delightful function, but I like my story.) You may suspect that this sort of procedure will take place again and again, as we seek compact notation for the functions that "solve" other important differential equations. Indeed it does! We have Legendre polynomials, various Bessel functions, spherical harmonics and many other "named functions" for just this purpose. But — pleasant surprise! — we can get by with *just the ones we have so far* for almost all of Newtonian Mechanics, provided we allow just one more little "extension" of the *exponential* function....

**Frequency = Imaginary Rate?**

Suppose we have

$$q(t) = q_0 e^{\lambda t}.$$

It is easy to take the $n^{\text{th}}$ time derivative of this function — we just "pull out a factor $\lambda$" $n$ times. For $n = 2$ we get $\ddot{q} = \lambda^2 q_0 e^{\lambda t}$ or just

$$\ddot{q} = \lambda^2 q. \tag{10}$$

Now go back to the example "solution" in Eq. (5), which turned out to be equivalent to HOOKES'S LAW [Eq. (6)]: $\ddot{x} = -\omega^2 x$, where $\omega = \sqrt{k/m}$ and $k$ and $m$ are the "spring constant" and the mass, respectively.

Equations (10) and (6) would be the *same equation* if only we could let $q \equiv x$ and $\lambda^2 = -\omega^2$. Unfortunately, there is no real number whose square is negative. Too bad. It would be *awfully nice* if we could just re-use that familiar exponential function to solve mass-on-a-spring problems too.... If we just use a little *imagination*, maybe we *can* find a $\lambda$ whose square is negative. This would require having a number whose square is $-1$, which takes so much imagination that we might as well call it $i$. If there were such a number, then we could just write

$$\lambda = i\omega. \tag{11}$$

That is, the *rate* $\lambda$ in the *exponential* formula would have to be an "*imaginary*" version of the *frequency* $\omega$ in the *oscillatory* version, which would mean (if the solution is to be *unique*) that

$$e^{i\omega t} = \cos \omega t.$$

It's not.

Oh well, maybe later....

## 12.2 Mind Your $p$'s and $q$'s!

Earlier we introduced the notion of *canonical coordinates* $q_i$ and the *generalized forces* $Q_i$ defined by the partial derivatives of the *potential energy* $V$ with respect to $q_i$. I promised then that I would describe a more general prescription later. Well, here it comes!

If we may assume that *both* the *potential energy* $V(q_i, \dot{q}_i)$ and the *kinetic energy* $T(q_i, \dot{q}_i)$ are known as explicit functions of the canonical coordinates $q_i$ and the associated "canonical velocities" $\dot{q}_i$, then it is useful to define the **Lagrangian** function

$$\mathcal{L}(q_i, \dot{q}_i) \equiv T(q_i, \dot{q}_i) - V(q_i, \dot{q}_i) \tag{12}$$

in terms of which we can then define the *canonical momenta*

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{13}$$

These canonical momenta are then guaranteed to "act like" the conventional momentum $m\dot{x}$ in all respects, though they may *be* something entirely different.

How do we obtain the equations of motion in this new "all-canonical" formulation? Well, HAMILTON'S PRINCIPLE declares that the motion of the system will follow the *path* $q_i(t)$ for which the "path integral" of $\mathcal{L}$ from initial time $t_1$ to final time $t_2$,

$$\mathcal{I} = \int_{t_1}^{t_2} \mathcal{L}\, dt \tag{14}$$

is an *extremum* [either a maximum or a minimum]. There is a very powerful branch of mathematics called the *calculus of variations* that allows this principle to be used[5] to derive the LAGRANGIAN EQUATIONS OF MOTION,

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} \tag{15}$$

Because the "$q$" and "$p$" notation is always used in advanced Classical Mechanics courses

[5] Relax, we aren't going to do it here.

to introduce the ideas of canonical equations of motion, almost every Physicist attaches special meaning to the phrase, "Mind your $p$'s and $q$'s." Now you know this bit of jargon and can impress Physicist friends at cocktail parties. More importantly, you have an explicit prescription for determining the equations of motion of *any* system for which you are able to formulate analogues of the potential energy $V$ and the kinetic energy $T$.

There is one last twist to this canonical business that bears upon greater things to come. That is the procedure by which the description is re-cast in a form which depends explicitly upon $q_i$ and $p_i$, rather than upon $q_i$ and $\dot{q}_i$. It turns out that if we define the **Hamiltonian** function

$$\mathcal{H}(q_i, p_i) \equiv \sum_i \dot{q}_i \, p_i - \mathcal{L}(q_i, \dot{q}_i) \tag{16}$$

then it is usually true that

$$\mathcal{H} = T + V \tag{17}$$

– that is, the Hamiltonian is equal to the *total energy* of the system! In this case the equations of motion take the form

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{18}$$

So what? Well, we aren't going to crank out any examples, but the Lagrangian and/or Hamiltonian formulations of Classical Mechanics are very elegant (and convenient!) generalizations that let us generate equations of motion for problems in which they are by no means self-evident. This is especially useful in solving complicated problems involving the rotation of rigid bodies or other problems where the motion is partially *constrained* by some mechanism [usually an actual machine of some sort]. It should also be useful to *you*, should you ever decide to apply the paradigms of Classical Mechanics to some "totally inappropriate" phenomenon like economics or psychology. First, however, you must invent analogues of *kinetic energy* $V$ and *potential energy* $T$ and give formulae for how they depend upon your canonical coordinates and velocities or momenta.

Note the dramatic paradigm shift from the *force* and *mass* of Newton's SECOND LAW to a complete derivation in terms of *energy* in "modern" Classical Mechanics. It turns out that this shift transfers smoothly into the not-so-classical realm of QUANTUM MECHANICS, where the HAMILTONIAN $\mathcal{H}$ takes on a whole new meaning.