THE UNIVERSITY OF BRITISH COLUMBIA

Physics 108

Sessional Examination

SOLUTIONS

15 April 2003

TIME: $2\frac{1}{2}$ HOURS

INSTRUCTOR(S): JESS H. BREWER

---

1. "QUICKIES" [40 marks — 5 each]

(*a*) Some butterflies have bright blue wings without any blue pigments. Explain briefly how this is possible. [1]

(*b*) To which of the laws represented by Maxwell's Equations did Maxwell himself actually make a direct contribution, and what was his contribution? [2]

(*c*) If ϵ_0 and μ_0 are respectively the permittivity and the permeability of free space, what is the value (including *units*) of the quantity $(\mu_0\, \epsilon_0)^{-1/2}$ and *what does it represent*? [3]

(*d*) In an LCR circuit driven by an AC voltage, the real current through the resistor is zero at any instant when [encircle *all* correct answers] [4]

i) the charge on the capacitor is zero.

ii) the charge on the capacitor is largest.

iii) the voltage across the inductor is zero.

iv) the voltage across the inductor is largest.

(*e*) In a hydrogen atom, the electrostatic force between the proton and electron is 2.3×10^{39} times greater than the gravitational force. If we can adjust the distance between the two particles, at what separation will the electrostatic and gravitational forces between them be equal in magnitude? *Explain.* [5]

(*f*) A capacitor is made from two metal plates of area A separated by a distance d. The gap is filled with two dielectric slabs of equal thickness but with different dielectric constants κ_1 and κ_2. Calculate the *capacitance* in terms of the parameters given. [6]

---

[1] By thin film interference. If the wing is coated with a film of higher index of refraction than the wing material and a thickness equal to one fourth of the wavelength of blue light (at that index of refraction) then blue light will be preferentially reflected relative to other colours. If the film has a *lower* index than the wing material, the same effect can be achieved if its thickness is *half* the wavelength, because in that case both reflected rays have the same phase reversal so the net path length difference needs to be a full wavelength. (I gave full credit for the first sentence of this answer.)

[2] Maxwell proposed the modification of Ampère's Law by adding the "displacement current" (rate of change of the electric flux through the loop) to the charge-flow current linking the loop.

[3] The speed of light, $c = 1/\sqrt{\mu_0\, \epsilon_0} = 2.99792458 \times 10^8$ m/s. You can work this out from the given values of the two constants, but it's a lot quicker to just remember this important fact.

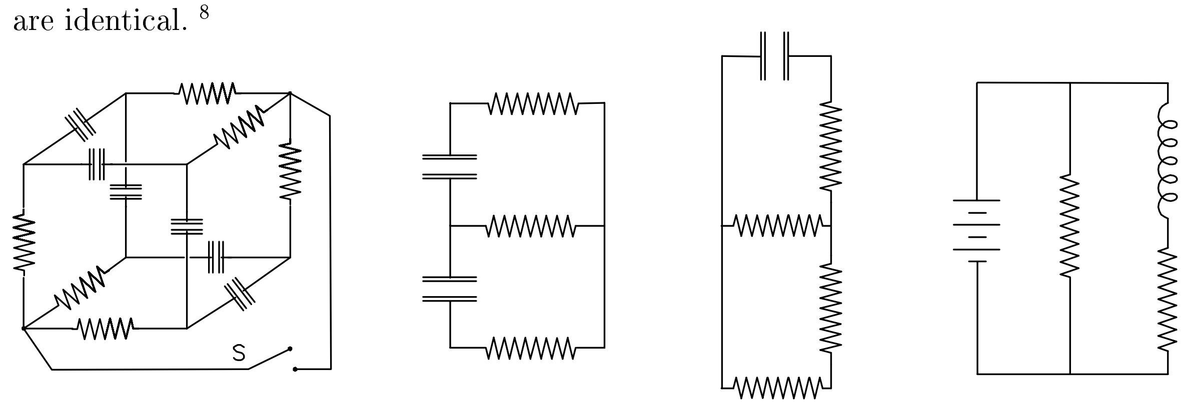
[4] The second and fourth answers are correct.

[5] They can never be equal. Both forces have the same dependence on distance, so the ratio of their strengths will always be the same. (This was a trick question designed to catch people who start calculating before they think.)

[6] If the plates carry a charge Q then the surface charge density is $\sigma = Q/A$. The field between the plates would thus be $E = \sigma/\epsilon_\circ = Q/\epsilon_\circ A$ if the gap were empty. The effect of the dielectric is to reduce E by a factor $\epsilon_\circ/\epsilon$. Thus in region 1 the field is $E_1 = Q/\epsilon_1 A$ and in region 2 the field is $E_2 = Q/\epsilon_2 A$. Moving from one plate to the other we travel a distance $d/2$

(g) Suppose you have a superconducting coil, a capacitor and a 1000 Ω resistor. First you charge up the capacitor and put it in series with the coil; this circuit oscillates at a frequency of $f = (1/2\pi)$ kHz. Next you charge up the same capacitor and discharge it through the resistor. (No coil in the circuit this time.) It takes 0.1 s for the charge on the capacitor to drop to $1/e$ of its original value. *What was the inductance of the coil?* [7]

(h) Encircle any of the following circuits which are "*simple*" (can be reduced to an equivalent series R and/or C and/or L and/or $\mathcal{E}$ circuit). The wires in the leftmost circuit are along the edges of a 3-dimensional cube. Assume that all similar circuit elements are identical. [8]



2. TWO-STATE SYSTEM [12 marks] A simple organism has only two possible microstates: "asleep" with zero energy or "awake" with energy $\varepsilon = 0.01$ eV. It is in thermal equilibrium with its environment.

   (a) [3 marks] At what *temperature* is the organism $1/e$ times as likely to be awake as to be asleep? ($e = 2.718\cdots$) [9]

   (b) [3 marks] At what temperature does it have an *equal* probability of being asleep or

along E_1 and another distance $d/2$ along E_2 for a net potential change of $V = E_1 d/2 + E_2 d/2 = (Qd/2A)(1/\epsilon_1 + 1/\epsilon_2)$. Since C is defined by $Q = CV$ or $C = Q/V$, we have $\boxed{C = \dfrac{2A}{\left[\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right] d}}$.

[7] The resonant frequency of the LC circuit is $1/\sqrt{LC} = \omega = 2\pi f = 10^3$ s^{-1} so $LC = 10^{-6}$ s^2. The time constant of the RC circuit is $\tau = RC = 0.1$ s for $R = 10^3$ Ω so $10^3 C = 0.1$ or $C = 10^{-4}$ F [Farads]. Thus $10^{-4} L = 10^{-6}$ or $\boxed{L = 0.01 \text{ H}}$ [Henries].

[8] The third circuit is simple because the array of resistors can be replaced by a single equivalent resistor. The fourth (rightmost) circuit is obviously *not* simple because it has unlike components in parallel; this can also be seen by picturing what it will "do" when the circuit is initially completed. The second circuit looks topologically suspicious, and would *not* be "simple" if the elements of the same type (resistors or capacitors) were not identical; but since both capacitors must have the same charge at all times and since the discharge path for each is also identical, a simple RC circuit with one equivalent capacitor and one equivalent resistor would have the same behaviour. The same is true of the first (leftmost) circuit for the same reasons, although it might be tedious to calculate the *values* of the equivalent R and C; in this case one must have a good eye for *symmetry* to see that each corner has three identical resistors in parallel, each of which collects charge from two identical capacitors. Again, if the symmetry were broken (*e.g.* by having one capacitor or resistor different from the others) the circuit would no longer be simple.

[9] Boltzmann showed that the probability a system being in a state of energy ε while in thermal equilibrium with a heat reservoir at temperature τ is proportional to $e^{-\varepsilon/\tau}$. For the state with zero energy this is just 1, so the relative probability of being in the other state is just $e^{-\varepsilon/\tau}$. If we want this to equal e^{-1} we must have $\tau = \varepsilon = 0.01$ eV $= 0.01 \times (k_B \times 11,600 \text{ K}) = k_B T$ or $T = 116$ K.

awake? [10]

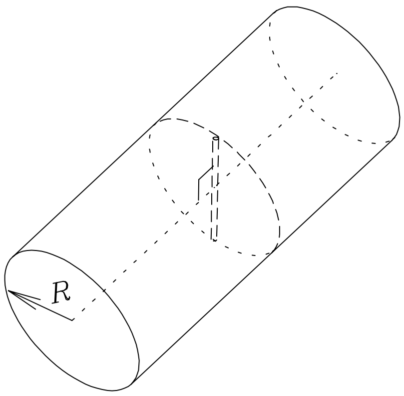
In the space below, *sketch*

(*c*) [3 marks] The probability of being awake as a function of $\tau \equiv k_{\rm B}T$ for a fixed "waking" microstate energy ε. [11]

(*d*) [3 marks] The probability of being awake as a function of the "waking" microstate energy ε for a given τ. [12]

(Include axis labels and vertical scales on your sketches.)

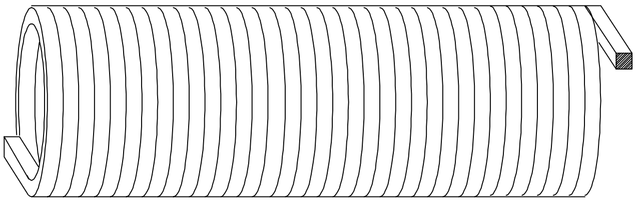
3. CYLINDER OF CHARGE [12 marks] A long solid insulating cylinder of radius R is uniformly charged with a positive charge density ρ [charge per unit volume]. A small hole is drilled straight through the cylinder at right angles to its axis, as shown. Assume that the hole is very far from the ends of the cylinder.



(*a*) [6 marks] Show that the electric field in the hole is proportional to the distance from the cylinder axis. [13]

(*b*) [6 marks] Show that a negatively charged particle dropped straight into the hole will execute simple harmonic motion as long as it doesn't hit the sides of the hole. [14]

4. SOLENOID DESIGN [12 marks]



[10]Following the same argument as before, we are looking for $e^{-\varepsilon/\tau} = 1$. This is only true if $\varepsilon/\tau = 0$, which for finite ε is only true when $\tau \to \infty$. You don't really have to do the mathematics to realize this, though, if you think about what the Boltzmann distribution describes. Anyone who got stuck on this part ought to have gotten a good retroactive hint from the graphs in the next part!

[11]The probability at $\tau = 0$ is zero; it stays "pretty small" until it shoots up around $\tau \approx \varepsilon$ and then slowly approaches 1/2 as $\tau \to \infty$. Don't forget that factor of 2! The best you can do it to make it *equally* probable to be either asleep or awake. (See previous question.)

[12]This is just an exponential decay from 1/2 at $\varepsilon = 0$ to zero as $\varepsilon \to \infty$, with a value of $\frac{1}{1+e}$ at $\varepsilon = \tau$.

[13]Imagine a cylindrical Gaussian surface of length L centred on the same axis, with radius $r < R$. The electric field is normal to the curved surface and has a constant magnitude everywhere on that surface (by symmetry) so the surface integral in Gauss' Law has the value $2\pi r L E$. The enclosed charge is $\pi r^2 L \rho$ so Gauss' Law gives $\boxed{E = (\rho/2\epsilon_0)\, r}$. *QED*.

[14]The electric field points away from the axis, so the force on a (negatively charged) electron points back toward the axis and has a magnitude proportional to the distance from the axis. This constitutes a linear restoring force and the electron has mass (inertia), so simple harmonic motion is automatic. *QED*

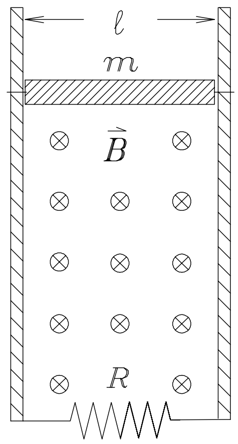
You are given a certain *volume* V of copper from which to make *wire* with a square cross section. The wire you make is then used to wind a *one-layer solenoid* as shown. (Ignore the practical necessity for insulation between the wires and pretend that they can be wound so that the distance between turns is exactly the width of the wire.)

Your goal is to generate the highest possible magnetic field at the centre of the solenoid.

(a) [6 marks] For wire of a given cross section and for a given *current* I in the wire, should you make the *inside diameter* of the solenoid large or small? Explain. [15]

(b) [6 marks] For a given inside diameter and for a given *voltage* $\mathcal{E}$ applied to the solenoid, should you make the wire *long and thin* or *short and thick*? Explain. [16]

5. FALLING BAR [12 marks]



A horizontal bar of mass m is free to slide without friction down the vertical rails of a conducting frame, as shown. The combined resistance of the bar and the frame is negligible compared to R, the resistance placed in series with this circuit. What is the *terminal speed* of the bar as it falls under the influence of gravity (at the surface of the Earth) through a uniform horizontal magnetic field $\vec{\boldsymbol{B}}$ directed perpendicular to the plane of the frame? (Give your answer in terms of m, B, ℓ, R, g and other fundamental constants.) [17]

---

[15] Let r be the radius of the solenoid and d be the width of the wire. The length of the solenoid is $\ell = Nd$ where N is the number of turns. $B = \mu_0 n I$ where $n =$ turns/unit length $= N/\ell = 1/d$, so for a given I, B is independent of the diameter $2r$, *as long as* there is enough wire to make the solenoid "long" ($\ell \gg r$). Otherwise the Ampère's Law formula for B will be an overestimate and the true B will be smaller. Thus a strictly correct answer would be, "*All other things being equal, smaller is slightly better.*"

[16] $I = \mathcal{E}/R$ and $R = \rho\ell/d^2$, so $I = \mathcal{E}d^2/\rho N d = \mathcal{E}d/\rho N$. [1] But for a given $V = Ld^2$ (where L is the total length of the *wire*) the number of turns is $N = L/2\pi r$ or $N = V/2\pi r d^2$. Put this back into [1] to get $I = \dfrac{\mathcal{E}}{\rho} \cdot \dfrac{2\pi r d^3}{V}$ and so $B = \mu_0 I \dfrac{N}{\ell} = \mu_0 \dfrac{I}{d} = \left(\dfrac{2\pi\mu_0\mathcal{E}r}{\rho V}\right) d^2$ implying that *short and thick is lots better*, until you start to have ℓ comparable to r, at which point you eventually reach the point of diminishing returns. Also, of course, you can't make $\ell < 2\pi r$ or you won't even be able to make up a single turn! There is another practical limit to this strategy: it is actually difficult to produce enormous currents at moderate voltage because any actual power supply has a resistance of its own, usually higher than that of a putative single turn of copper bar. The would-be Electrical Engineer may want to perform a calculation of the optimum conductor dimensions to generate the maximum B with the least possible *power* consumption, given a finite resistance in the power supply itself. . . .

[17] As the bar drops at a speed v, the magnetic flux through the loop formed by the bar, rails and resistor decreases at a rate $d\Phi_M/dt = \ell v B$, giving an induced EMF V of the same magnitude. By Lenz' Law this EMF will "try" to

6. 3-SLIT GRATING [12 marks] Most gratings have a very large number of "slits" — but picture one that has only *three* 25 μm wide slits, uniformly spaced 100 μm apart (center to center). This grating is uniformly illuminated with greenish blue light of wavelength $\lambda = 500$ nm incident normal to the plane of the grating, producing an interference pattern on a screen 1 m away, oriented parallel to the plane of the grating.

(*a*) [4 marks] *Make a detailed sketch of the resulting interference pattern* (intensity as a function of position on the screen). You may plot the intensity in arbitrary units, but please include units and dimensions on the position axis. Show at least the range from the central maximum out to the first diffraction minimum on each side. [18]

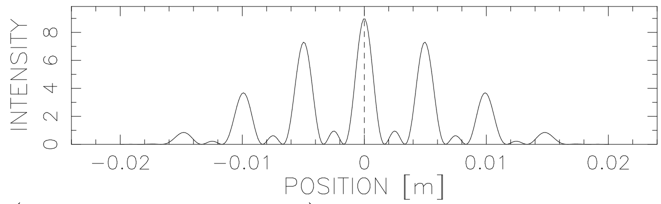
(*b*) [4 marks] Ignoring diffraction due to the finite width of the individual slits, *sketch the phasor diagram* for the position in the interference pattern where the *intensity* first drops to one ninth (**1/9**) of its value at the central maximum. [19]

(*c*) [4 marks] Suppose that light of a slightly different colour (wavelength $\lambda + \Delta\lambda$) shines on the grating in the same way at the same time. What is the *smallest* wavelength difference $\Delta\lambda$ that can just barely be resolved at the first principal maximum? [20]

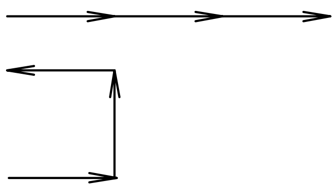
---

replace the flux into the page: a current will be induced in such a direction as to make its own magnetic field into the page, namely *clockwise*. The resulting current will be $I = V/R = \ell vB/R$. A current of this size in a perpendicular magnetic field B produces a force $F = IB\ell$ on the bar. Checking with the right hand rule confirms that the force will be *up*, thus counteracting the downward force of gravity. The magnetic force increases as the velocity increases, until it exactly balances mg, at which velocity (v_t) there is no net force and *terminal velocity* will have been reached. Thus $(\ell v_t B/R)(B\ell) = mg$ or $\boxed{v_t = \dfrac{mgR}{B^2\ell^2}}$.

[18] The wide *diffraction* envelope from the narrow *slit width* has a first minimum at $\sin\vartheta_1^D = \lambda/a = 0.5/25 = 0.02$ whereas the m^{th} principal maximum of the *interference* pattern from the 3 slits is at $\sin\vartheta_m^I = m\lambda/d = m(0.5/100) = 0.005m$. The position x on a flat screen 1 m away is given by $x[\text{m}] = \tan\vartheta$. All the angles are small out to at least ϑ_1^D, so we can use the small angle approximations $\tan\vartheta \approx \sin\vartheta \approx \vartheta$ to get $x_1^D = 0.02$ m (2 cm) and $x_m^I = 0.005m$ m ($0.5m$ cm). Note that x_1^D is right on top of x_4^I and the principal maxima are evenly spaced. The only other thing you need to remember to get the full pattern is that for $N = 3$ slits there are $N - 1 = 2$ minima and $N - 2 = 1$ secondary maximum between the principal maxima. The pattern therefore looks like this:



[19] For the central maximum (or any principal maximum) the phasor diagram is three vectors of the same length added tip-to-tail all in the same direction, so the net amplitude is three times that of each. When the three individual phasors are perpendicular as shown, the net amplitude is just the missing side of the square, or 1/3 as big as for the central maximum. Since the intensity is the *square of the amplitude*, this gives 1/9 of the central intensity.



[20] *Rayleigh's criterion* says that two lines are just resolvable if the principal maximum of one falls right on top of the first minimum beyond that principal maximum of the other. A grating gives larger *dispersion* (change of angle with wavelength) for higher order principal maxima, but here we are using $m = 1$. Thus the criterion is that $\vartheta_1^I(\lambda + \Delta\lambda)$, the angle of the first principal maximum for the second wavelength, is equal to the angle $\vartheta_{1+1/N}^I(\lambda)$ of the first local minimum beyond the first principal maximum for the original wavelength. That this occurs $1/N = 1/3$ of the way to the second

principal maximum can be seen by picturing the phasor diagram for that first minimum: an equilateral triangle with each phasor $\delta = 2\pi(1 + 1/3) = 360^\circ + 120^\circ$ out of phase with the next. Since $\delta/2\pi = (d/\lambda)\sin\vartheta \approx d\vartheta/\lambda$, this value of δ corresponds to $\vartheta = (1 + 1/3)\lambda/d$. At the same angle we have the first principal maximum ($\delta' = 2\pi$) for $\lambda + \Delta\lambda$, that is, $\vartheta = (\lambda + \Delta\lambda)/d$. Combining these results gives $(1+1/3)\lambda = \lambda + \Delta\lambda$ or $\boxed{\Delta\lambda = \frac{\lambda}{3} = 166.7 \text{ nm}}$. Note that you could get the same answer by just plugging into the formula for the *resolving power* of a grating at the m^{th} principal maximum, but that wouldn't be nearly as much fun!