RELATIVISTIC KINEMATICS (Mass is Energy is Mass)

Since **Mechanics** is so intimately concerned with the relationships between *mass*, *time* and *distance*, the weird properties of the time and space revealed by the $\mathcal{STR}$ may be expected to be accompanied by some equally weird **Mechanics** at relativistic velocities. This is indeed the case. On the other hand, we can rely upon Einstein's first postulate of the $\mathcal{STR}$, namely that the "Laws of Physics" are the same in one reference frame as in another. Thus most of our precious paradigms of **Mechanics** (such as **conservation laws**) will still be reliable.

$E{=}mc^2$.1 Momentum is Still Conserved

For instance, **momentum conservation** must still hold, or else we would be able to tell one reference frame from another (in an *absolute* sense) by seeing which one got less than its share of momentum in a collision. To pursue this example, we invoke **momentum conservation** in a *glancing collision* between two identical billiard balls, as pictured below.

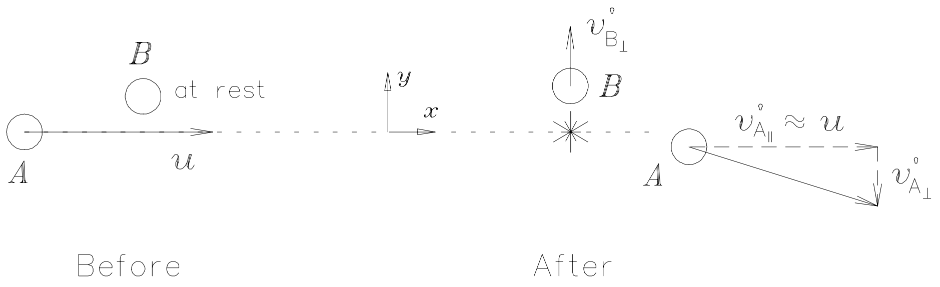

Figure 1: A *glancing collision* between two identical billiard balls of rest mass m. Ball A barely touches ball B as it passes at velocity u, imparting a miniscule *transverse* velocity $v_{B_\perp}$ (perpendicular to the initial velocity of A) to ball B and picking up its own transverse velocity $v_{A_\perp}$ in the process. *Unprimed* quantities (like $v_{A_\perp}$ and $v_{B_\perp}$) are measured in A's reference frame, whereas *primed* quantities (like $v'_{A_\perp}$ and $v'_{B_\perp}$) are measured in B's reference frame. This drawing is shown in B's reference frame.

Now, each of A and B measures[1] its *own* final transverse velocity (relative to the fixed dotted line along which A was originally travelling) and sends a message to the other describing the results. These messages will agree — *i.e.*, $v_{A_\perp} = v'_{B_\perp}$ — but if B measures the final transverse velocity of A in *its* (B's) rest frame, namely $v'_{A_\perp}$, the result will be different due to **time dilation**. The logic of this statement is as follows: $v'_{A_\perp} = \Delta y'_A / \Delta t'_A$ where $\Delta y'_A$

[1] If the anthropomorphism of billiard balls bothers you, please imagine that these are *very large* "billiard balls" with cabins occupied by Physicists who make all these observations and calculations.

is the distance travelled by A (perpendicular to its original direction) in a time $\Delta t'_A$ (as seen by B). Now, $\Delta y'_A = \Delta y_A$ because lengths transverse to the direction of motion do not differ between reference frames. On the other hand, $\Delta t'_A = \gamma \Delta t_A$ because of **time dilation** — the nearly stationary observer on B measures a longer time interval for processes in the moving frame A than are observed by A itself. Since $v_{A_\perp} = \Delta y_A / \Delta t_A$ in A's own frame, this means B "sees" a *smaller* transverse velocity $v'_{A_\perp}$ for A than A "sees" in its own rest frame ($v_{A_\perp}$), by a factor of $1/\gamma$:

$$v'_{A_\perp} \; = \; \frac{1}{\gamma} \, v_{A_\perp} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \text{and} \qquad \beta = \frac{u}{c}. \tag{1}$$

Meanwhile we must have **momentum conservation** for the collision. This means that the transverse momentum of A as observed by B, namely $m'_A \, v'_{A_\perp}$, must be the same as that of B, namely $m'_B \, v'_{B_\perp}$. But the latter transverse velocity ($v'_{B_\perp}$) is the same as that of A as measured in its own frame ($v_{A_\perp}$) and the mass of either ball measured in its own frame is just the **rest mass** m, so we have

$$m'_A \, v'_{A_\perp} \; = \; m \, v_{A_\perp} \tag{2}$$

where you may have noticed that I have started putting subscripts and primes on *masses* in anticipation of the conclusion that the only way to reconcile **momentum conservation** [Eq. (2)] with the relativity of transverse velocity [Eq. (1)] is to treat the *mass* — or at least the *effective* mass — as yet another quantity which may have different values when viewed from different reference frames. The result is obtained easily from Eq. (2) and Eq. (1):

$$m' \; = \; \gamma \, m \tag{3}$$

where the "$_A$" subscript has been dropped to make Eq. (3) more general. In words, it reads

> *The* **effective mass** m' *of an object moving at a velocity* $u = \beta c$ *is* γ *times its* **rest mass** m *(its mass measured in its own rest frame).*

That is, *moving masses have more inertia!*

$E{=}mc^2$.1.1 Another Reason You Can't Go as Fast as Light

The preceding argument was not very rigourous, but it served to show the essential necessity for regarding the **effective mass** of an object as a *relative* quantity. Let's see what happens as we try to accelerate a mass to the velocity of light: at first it picks up speed just as we have been trained to expect by Galileo.[2] But as β becomes appreciable, we begin to see

[2]It had better! The behaviour of *slow-moving* objects did not undergo some sudden retroactive change the day Einstein wrote down these equations!

an interesting phenomenon: *it gets harder to accelerate!* (This is, after all, what we *mean* by "effective mass.") As $\beta \to 1$, the multiplicative "mass correction factor" $\gamma \to \infty$ and eventually we can't get any more speed out of it, we just keep pumping energy into the effective mass. This immediately suggests a new way of looking at mass and energy, to be developed in the following Section.

But first let's note an interesting side effect: the rate at which a constant accelerating force produces velocity changes, *as measured from a nonmoving reference frame*, slows down by a factor $1/\gamma$; but the *same* factor governs the **time dilation** of the "speed" of the clock in the moving frame. So (as observed from a stationary frame) the change in velocity *per tick of the clock* in the moving frame is constant. This has no practical consequences that I know of, but it is sort of cute.

$E=mc^2$.2 Mass and Energy

In the hand-waving spirit of the preceding Section, let's explore the consequences of Eq. (3). The **binomial expansion** of γ is

$$\gamma \;=\; (1-\beta^2)^{-\frac{1}{2}} \;=\; 1 \;+\; \tfrac{1}{2}\beta^2 \;-\; \cdots \tag{4}$$

For *small* β, we can take only the first two terms (later terms have still higher powers of $\beta \ll 1$ and are considered negligible) to give the approximation

$$m' \;\approx\; (1+\tfrac{1}{2}\beta^2)\,m \qquad \text{or} \qquad m'c^2 \;\approx\; mc^2 \;+\; \tfrac{1}{2}mu^2 \tag{5}$$

The last term on the right-hand side is what we ordinarily think of as the **kinetic energy** T. So we can write the equation (in the limit of small velocities) as

$$T \;=\; \gamma\, mc^2 \;-\; mc^2 \tag{6}$$

It turns out that Eq. (6) is the *exact* formula for the kinetic energy at *all* velocities, despite the "handwaving" character of the derivation shown here.

We can stop right there, if we like; but the two terms on the right-hand side of Eq. (6) look so simple and similar that it is hard to resist the urge to give them names and start thinking *in terms of them*.[3] It is conventional to call $\gamma\, mc^2$ the **total relativistic energy** and mc^2 the **rest mass energy**. What do these names mean? The suggestion is that there is an irreducible energy $E_0 = mc^2$ associated with any object of mass m, even when it is sitting still! When it speeds up, its total energy changes by a multiplicative factor γ; the *difference* between the total energy $E = \gamma\, mc^2$ and E_0 is the energy due to its *motion*, namely the *kinetic energy* T.

[3]This is, after all, the most ubiquitous instinct of Physicists and perhaps the main æsthetic foundation of Physics. It is certainly what I mean by "Physics as Poetry!"

$E{=}mc^2$.2.1 Conversion of Mass to Energy

Einstein's association of the term mc^2 with a **rest mass energy** E_0 naturally led to a great deal of speculation about what might be done to *convert* mass into useable energy, since for a *little* mass you get a *lot* of energy! Let's see just how much: in *S.I.* units $1 \text{ J} \equiv 1 \text{ kg-m}^2/\text{s}^2$ so a 1 kg mass has a rest mass energy of $(1 \text{ kg}) \times (2.9979 \times 10^8 \text{ m/s})^2 = 8.9876 \times 10^{16}$ J — *i.e.*,

$$1 \text{ kg} \longleftrightarrow 8.9876 \times 10^{16} \text{ J} \tag{7}$$

which is a lot of joules. To get an idea how many, remember that one **watt** is a unit of *power* equal to one joule per second, so a **joule** is the same thing as a **watt-second**. Therefore a device converting *one millionth of a gram* (1 μg) of mass to energy *every second* would release approximately *90 megawatts* [millions of watts] of power!

Contrary to popular belief, the first conclusive demonstration of mass-energy conversion was in a controlled nuclear *reactor*. However, not long after came the more unpleasant manifestation of mass→energy conversion: the fission bomb. An unpleasant subject, but one about which it behooves us to be knowledgeable. For this, we need a new energy unit, namely the **kiloton** [kt], referring to the energy released in the explosion of one thousand *tons* of TNT [*tri**n**itro**t**oluene], a common chemical high explosive. The basic conversion factor is

$$1 \text{ kt} \equiv \text{ a trillion } \textbf{calories} = 4.186 \times 10^{12} \text{ J} \tag{8}$$

which, combined with Eq. (7), gives a rest-mass equivalent of

$$1 \text{ kt} \longleftrightarrow 4.658 \times 10^{-5} \text{ kg} \tag{9}$$

That is, one **kiloton**'s worth of energy is released in the conversion of 0.04658 grams [46.58 mg] of mass. Thus a **megaton** [equivalent to one million tons of TNT or 10^3 kt] is released in the conversion of 46.58 grams of mass; and the largest thermonuclear device [bomb] ever detonated, about 100 megatons' worth, converted some 4.658 kg of mass directly into raw energy.

Nuclear Fission

Where did the energy come from? *What* mass got converted? To answer this question we must look at the processes involved on a sub-microscopic scale. First we must consider the natural tendency for oversized atomic nuclei to spontaneously *split* into smaller components.[4] This process is known as **nuclear fission** and is the energy source for all presently functioning **nuclear reactors** on Earth. [Also for so-called "atomic" bombs.][5]

[4] I know I haven't explained what I mean by a "nucleus" yet, or even an "atom;" but here I will suspend rigourous sequence and "preview" this subject. The details are not important for this description.

[5] The name, "**atomic bomb**," is a frightful misnomer; the *atoms* have nothing whatsoever to do with the process involved in such horrible weapons of destruction, except insofar as their *nuclei* are the active ingredients. The correct name for the "atomic" bomb is the **nuclear fission** bomb.

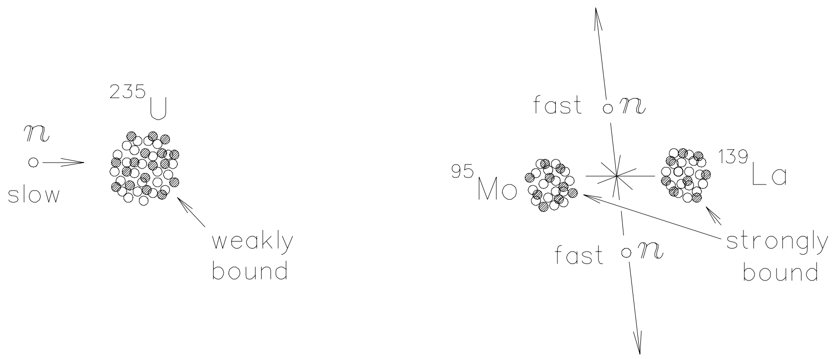

Figure 2: One case of the fission of ^{236}U. The net mass of the initial neutron plus the ^{235}U nucleus is 219,883 MeV/c^2. The net mass of the fission products (two neutrons, a ^{95}Mo nucleus and a ^{139}La nucleus) is 219,675 MeV/c^2 — smaller because of the stronger *binding* of the Mo and La nuclei. The "missing mass" of 208 MeV/c^2 goes into the *kinetic energy* of the *fragments* (mainly the *neutrons*), which of course adds up to 208 MeV.

The basic event in the most common variety of **nuclear fission** is the spontaneous splitting of one ^{236}U nucleus into (for example) ^{95}Mo, ^{139}La and two neutrons.[6] [There are numerous other possible fission products. This is just one case.] The fraction of the total mass that gets converted into kinetic energy is $208/219833 = 0.946 \times 10^{-3}$ or about a tenth of a percent. The energy liberated in the fission of one ^{236}U nucleus produced in this way is 208 MeV or 0.333×10^{-4} J. That means it takes 30,000 such fissions to produce one joule of utilizable energy. Since there are 2.55×10^{24} such nuclei in one gram of pure ^{235}U metal, 30,000 isn't such a large number!

What sort of *control* do we have over this process? To answer this question we must understand a bit more about the details of the **chain reaction** whereby an appreciable number of such fissions take place.

The ^{236}U nucleus is formed by adding one neutron to a ^{235}U nucleus, which is found in natural uranium ore on Earth at a concentration of about 0.72% [the rest is almost all ^{238}U]. Now, left to its own devices (*i.e.*, if we don't drop any slow neutrons into it) a ^{235}U nucleus

[6] The notation used here is ^{A}El, where the *atomic weight* A of an element is the total number of *neutrons* (uncharged nucleons) and *protons* (positively charged nucleons) in the nucleus and *El* is the *chemical symbol* for the *element* in question. "Nucleon" is just a generic name for either protons or neutrons, which have about the same mass [the neutron is slightly heavier] and the number of *protons* in a nucleus [called its *atomic number* Z] determines its net electrical charge, which in turn must be balanced by an equal number of negatively charged *electrons* in orbit about the nucleus to make up the *atom*. The *atomic number* Z therefore determines all the *chemical* properties of the atom and so defines which *element* it is. We could just specify Z in addition to A to know everything we need to know about the specific nucleus in question [which we call an **isotope**], but names are more appealing than numbers [even to Physicists!] so we use the chemical symbol [*e.g.*, U = Uranium, Mo = Molybdenum, La = Lanthanum, H = Hydrogen, He = Helium and Li = Lithium] as an abbreviation for the name of the element. Sometimes you will see Z as a *subscript* on the left of the chemical symbol, as in $^{238}_{92}$U, but this is not the only convention for isotopic notation and I see no reason to confuse matters any further. There — a micro-introduction to nuclear, atomic and chemical terminology!

will live for an average of 0.7038 billion years, eventually decaying spontaneously by α particle emission (*not* the fission reaction that produces more neutrons!) just like its brother isotope ^{238}U, whose lifetime is only about 6 times longer (4.468 billion years). If the lifetimes weren't so long, there wouldn't be any left on Earth to dig up — which might be regarded as a good thing overall, but we have to play the hand we're dealt. So an isolated ^{235}U nucleus generally sits around doing nothing and minding its own business; but when a slow *neutron* comes by (picture a ball bearing slowly rattling down through a peg board) it has a strong tendency to be *captured* by the ^{235}U nucleus to form ^{236}U, and then the action starts. This is also a little tricky, because if the ^{236}U nucleus gets a chance to settle into its *ground state* (*i.e.*, if all the jiggling and vibrating caused by absorption of a neutron has a chance to die down) then it (the ^{236}U nucleus) is also quite stable [mean lifetime = 23.42 million years] and also decays by α emission (no new neutrons). However, this is rarely the case; usually the excitations caused by absorbing that extra neutron are too much for the excited ^{236}U nucleus and it *fissions* as described earlier, releasing several not-too-fast neutrons.

What follows depends upon the neighbourbood in which the fission occurs. If the original ^{235}U nucleus is off by itself somewhere, the two neutrons just escape, rattle around until they lose enough energy to be captured by some less unstable nuclei, and the process ends. If the fission occurs right next to some *other* ^{235}U nuclei, then the outcome depends (critically!) upon the **moderation** [slowing down] of the neutrons: when they are emitted in the fission process, they are much too fast to be captured by other ^{235}U nuclei and will just escape to bury themselves eventually in some innocuous nuclei ensewhere. If, however, we run them through some **moderator** [slower-downer] such as graphite, heavy water (deuterium oxide, D_2O) or, under extreme conditions of density and pressure, uranium metal itself, the neutrons will slow down by a sort of frictional drag until they reach the right energy to be captured efficiently by other ^{235}U nuclei. Then we get what is known as a **chain reaction**. One neutron is captured by a ^{235}U nucleus which splits up into fission products including fast neutrons, which are moderated until they can be captured by other ^{235}U nuclei, which then split up into fission products including fast neutrons, which are. . . .

The moderation of the neutrons generates a lot of *heat* in the moderator (it is a sort of *friction*, after all) which can be used in turn to boil water to run steam turbines to generate electricity. [Or misused to make a large explosion.] A good fission **reactor** design (like the Canadian CANDU reactor) involves a moderator like heavy water (D_2O) which boils away when the reactor core overheats, thus stopping the moderation and automatically shutting down the reactor. A bad design (like the Soviet or American reactors) uses **moderator rods** that are shoved into the core mechanically and can get stuck there if the core overheats, as happened at Three Mile Island and (much worse) at Chernobyl.[7]

[7]There is an interesting history to the American [and presumably the Soviet] reactor design: the original version was built on a small scale to go into nuclear submarines, where it worked quite well (and was comparatively safe, considering the unlimited supply of coolant!). However, the successful submarine reactor design was simply *scaled up* to make the big land-based power reactors, a thoroughly dumb and lazy manœuver by the power industry that has led to a long series of unnecessary troubles. If the world had standardized on the CANDU design, nuclear power would have a much better reputation today, except for the irreducible (though undeserved) taint of psychological association with nuclear weapons, which has even prompted doctors to change the name of NMR (nuclear magnetic resonance) imaging machines — probably the most harmless

Potential Energy is Mass, Too!

Where did the mass "go" in the reaction we just discussed? The answer is that the **binding energy** of the ^{235}U nucleus is substantially *less negative* than that of the final products.

Remember that the *gravitational potential energy* between two massive bodies is *zero* when they are infinitely far apart and becomes more and more *negative* as they get closer together? [Lower gravitational potential energy for an object at a lower height?] Well, the **strong nuclear force** that binds nuclei together has at least this much in common with gravity: it is *attractive* (at least at intermediate range) and therefore produces a **potential energy "well"** into which the constituents "fall" when we make up a nucleus.[8]

The other thing to realize is that *potential energy counts* in the evaluation of the total relativistic energy of an object; and if the object is at rest, then its potential energy counts in the evaluation of its **rest mass**. As a result, we might expect the rest mass of a space ship to be slightly larger after it leaves the Earth than it was on Earth, simply because it has left the "gravity well" of the Earth. This is the case! However, the mass change is imperceptibly *small* in this case.

Nuclear Fusion

Actually, a large nucleus is *rarely* heavier than the sum of its constituents. If you think about it, this is the equivalent of having a ball stored at the top of a potential energy *hill*.[9] Once it moves over the edge, the process is all downhill, resulting in liberation of kinetic energy. The heaviest nuclei represent *stored-up energy* from "endothermic" (energy-absorbing) processes that took place in **supernova** explosions billions of years ago, and are in that sense correctly referred to as "supernova fossils." Anything heavier than *iron* falls into this category!

Nuclei *lighter* than iron (^{57}Fe), if they can be regarded as composed of lighter nuclei, are almost always *lighter* than the sum of their constituents, simply because their **binding energy** is greater. The process of combining light nuclei to make heavier ones (up to iron) is called **nuclear fusion**, which also liberates kinetic energy. There are many, many varieties of nuclear fusion reactions, most of which are realized on a large scale in *stars*, whose main energy source is nuclear fusion. [A nice, romantic aspect of nuclear physics, for a change!] Our

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and beneficial devices ever created by modern technology — to "MRI" (for Magnetic Resonance Imaging) just so their patients wouldn't be spooked by the boogey-word "nuclear."

[8] Note how extensively we rely on this gravitational metaphor! This is partly because we don't know any more compelling poetic technique and partly because it works so well — it is a "good" metaphor!

[9] If you think about it some more, you will realize that such a situation usually constitutes **unstable equilibrium**: the tiniest push will set the ball rolling downhill, never to return of its own accord. In this case (carrying the nice metaphor a little further) there is actually a slight *depression* at the top of the hill, so that the ball can rest easy in **metastable equilibrium**: as long as it doesn't get to rolling around too energetically [enough to roll up over the edge of the depression], the ball will stay where it is; but if we "tickle" it enough [in this case, by dropping in a neutron] it will bounce out and from there it is all downhill again. This picture works almost perfectly in developing your intuition about metastable nuclei, except for the peculiar prediction of **quantum mechanics** that the ball can get through the "barrier" without ever having enough kinetic energy to make it up over the ridge! But that's another story....

own Sun, for example, is one big fusion power plant and has *all* the pleasant and unpleasant features of the putative man-made versions, such as radiation hazards. . . .

Unfortunately, here on Earth we have not yet succeeded in *controlling* **nuclear fusion** well enough to make a reactor that will generate more energy than it takes to run, though billions of dollars have been (and will doubtless continue to be) spent in the attempt. So far all we have achieved with notable success is the *uncontrolled* thermonuclear[10] reaction [bomb] known as the "H bomb."[11] A nasty feature of thermonuclear bombs is that there doesn't seem to be an upper limit on how big one might make them. The only good thing about them (other than the questionable virtue of "deterrence") is that they are not intrinsically as "dirty" (in terms of radioactive fallout) as fission bombs, at least not "per kiloton." But I have said rather more than I like about this subject already.

Cold Fusion

"Wouldn't it be nice," most reasonable people would agree, "if there were a way to obtain energy from fusion of some innocuous nuclei like deuterium without the enormous temperatures of nuclear explosions or the various 'hot' controlled fusion reactors on the drawing boards." There certainly is a way to get deuterium nuclei close enough together to fuse without high temperatures — in fact I recently participated in an experiment that achieved D-D fusion at a temperature of 2.5 K: this involves forming a *molecule* of two deuterons and one negative *muon* — an unstable elementary particle which is more or less like an electron except that its mass is 207 times bigger. The heavy muon pulls the deuterons so close together that they fuse. This works. Unfortunately *it doesn't work well enough to generate more energy than it took to make the muon in the first place!* The closest anyone has come to "break even" using muons is more than a factor of ten too low in efficiency. Too bad. It is frustrating to come so close and then fail.

Perhaps because of this frustration, a few people recently deluded themselves into believeing they had coaxed deuterons into fusing by regular electrochemical means in a palladium metal matrix. Unfortunately this was bogus, as numerous more careful tests clearly showed. However, the dream of unlimited free energy easily generated with simple equipment is so seductive that many "Cold Fusion" believers still hang onto it; moreover, they seem to have little trouble convincing sponsors to give them money to maintain their campaign to overcome experimental refutations with good press. For an interesting study in the sociology of science, see the *World Wide Web* home page

[10]We call such processes *thermo*nuclear because the positively charged nuclei don't "like" to get close enough to each other for the strong, short-range nuclear force to take over (they repell each other electrically), and to overcome this "Coulomb barrier" they are heated to such enormous temperatures that their kinetic energy is high enough to get them together and then . . . *bang!* The *heating* is usually done by means of a very small *fission* bomb, from what I understand.

[11]Once again, the popular terminology "H bomb" is completely misleading. The first thermonuclear bombs used a mixture of deuterium (^{2}H) and tritium (^{3}H) — two isotopes of hydrogen — as the components that fused to form heavier products, hence the name; but modern thermonuclear bombs use (I think) deuterium and lithium, which can be combined chemically into a solid form that is relatively easy to handle and not spontaneously radioactive.

`http://www.mit.edu:8001/people/rei/CFdir/CFhome.html` — for the other side of the story, talk to almost any physicist.

$E=mc^2$.2.2 Conversion of Energy into Mass

In a **nuclear reactor**, a spontaneous nuclear process results in a net *decrease* in the net mass of all the particles involved. The "missing mass" appears as the *kinetic energy* of the reaction products, which is dissipated by what amounts to friction and generates *heat* that boils water; the steam is used to spin turbines that run generators that send electrical power down the wires.

This leads to an obvious question: can we do the *opposite*? Can we take electrical power out of the wires, use it to raise the kinetic energy of some particles to enormous values, smack the particles together and *generate* some *extra* mass? Yes! This is what a **particle accelerator** like **triumf**[12] does. Every such accelerator is a sort of "reactor in reverse," taking electrical power out of the grid and turning it into mass.

Such things happen *naturally*, too. Gamma rays of sufficient energy often convert into electron-positron *pairs* when they have a glancing collision with a heavy nucleus. This is pictured below.

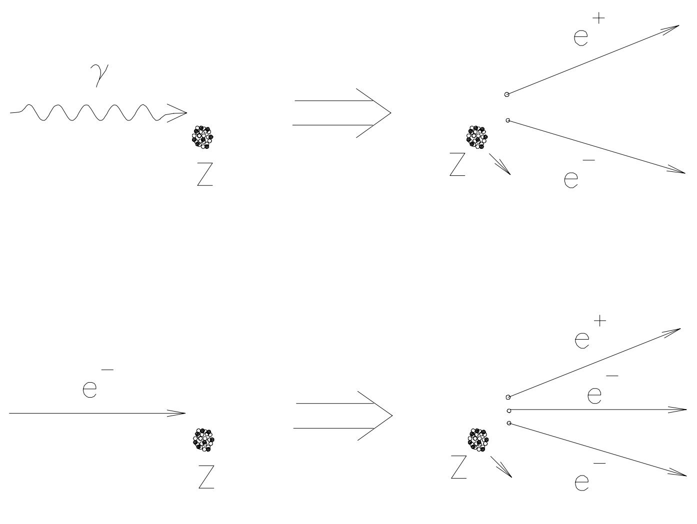

Figure 3: Electron-positron **pair production** by gamma rays (above) and by electrons (below). The positron (e^+) is the **antiparticle** of the electron (e^-) [its opposite in every way]. The gamma ray (γ) must have an energy of at least 1.022 MeV [twice the rest mass energy of an electron] and the pair production must take place near a heavy nucleus (Z) which absorbs the momentum of the γ.

[12] The acronym **triumf** stands for **tri-u**niversity **m**eson **f**acility, in recognition of the three B.C. Universities that originally founded to project [there are now several more, but we don't change the cute name] and the main product of the cyclotron.

There is a neat, compact way of representing such reactions by **Feynman diagrams**[13] I will draw them "left to right" but the convention is actually to draw them "down to up." I don't know why.

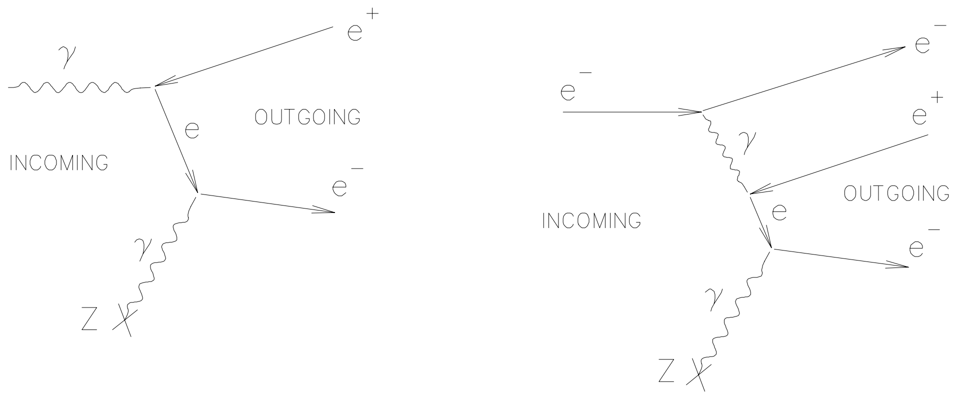

Figure 4: **Feynman diagrams** for pair production by a gamma ray (left) or an electron (right). These represent the processes in the preceding sketch.

The convention in **Feynman diagrams** is that *antiparticle* lines (e^+, for instance) are drawn in the "backward" sense as if they were propagating backward in time. This allows all "electron lines" to be *unbroken*, a graphical expression of the **conservation of electrons**.[14] There are lots more elegant graphical features to **Feynman diagrams**, but the details will have to wait for a course in **quantum field theory**.[15]

The main point here is that the *incoming* particle(s) [γ or e^-] must have at least 1.022 MeV of kinetic energy to create a positron and an electron, both of which have rest masses of 0.511 MeV/c^2. With an *accelerator* one can give the original projectile(s) more energy [there seems to be no limit on how much, except for mundane concerns about funding resources and real estate] and thus facilitiate the creation of *heavier* particles. At **triumf**, for instance, we accelerate protons to 520 MeV [just over half their rest mass energy of 938 MeV], which is enough to create π **mesons** [mass = 139 MeV/c^2] with reasonable efficiency; the high *intensity*[16] of the **triumf** cyclotron qualifies it for the elite club of "**meson factories**," so named because they "mass produce" π mesons (or **pions**) in unprecedented numbers.

Since heavier particles can in principle decay into lighter particles like gamma rays, neutrinos, antineutrinos, electrons and positrons, almost of these "manufactured" particles are unstable. Nevertheless, they hang around long enough to be studied and sometimes their

[13] This is basically what won Feynman his Nobel Prize; these simple diagrams are rigourously *equivalent* to great hairy *contour integrals* that you would not really want to see! Thus Feynman brought the Right Hemisphere to bear on elementary particle physics. Without this simple tool I wonder how far we would have come by now. . . .

[14] Note that *gamma* particles [photons] are *not* conserved — they are always being created or destroyed!

[15] Of course, if you are interested now and just can't wait, I can recommend some good reading. . . .

[16] The intensity of an accelerated particle beam can be measured in particles per unit time [**triumf** has about 10^{15} protons/sec] or, if the particles carry electric charge, in **amperes** of electrical current [**triumf** has about 140 μA (microamperes)].

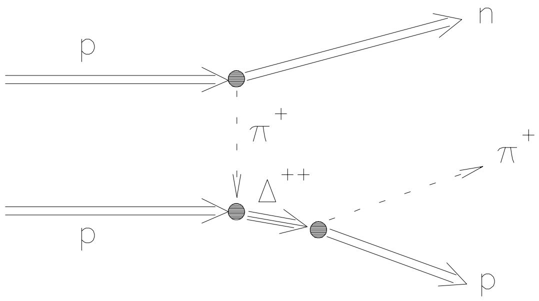

Figure 5: Feynman diagram for production of a π^+ meson by a collision between two protons (the most important interaction at **triumf**).

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very instability is what makes them interesting, if only because it precludes finding a cache of them in a more Natural setting.

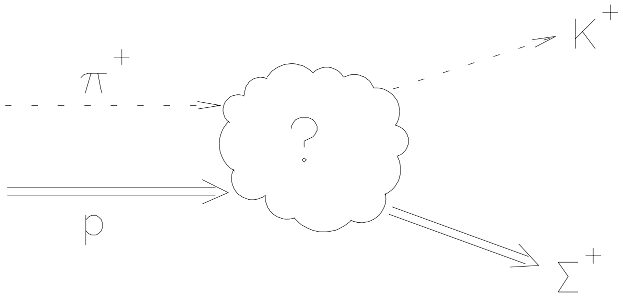

Figure 6: Feynman diagram for "**associated production**" of a K^+ meson [mass = 494 MeV/c^2 and "strangeness" $S = +1$] and a Σ^+ **hyperon** [a type of **baryon** with mass = 1193 MeV/c^2 and strangeness $S = -1$] in a collision between a π^+ and a proton (the pions produced at **triumf** don't have enough energy to do this).

I have gotten *far* beyond my terms of reference here, but I wanted to "preview" some of the phenomenology of Elementary Particle Physics while focussing your attention on the simple motive for building higher- and higher-energy accelerators:

> The more kinetic energy is available, the more mass can be created. The heavier the particle, the more options it is apt to have for other lighter particles to decay into, and the more unstable it can be expected to be; hence the less likely we are to observe it in Nature.[17] And the heavier the particle, the more exotic its properties might be.

So far this simple strategy has paid off in many new discoveries; of course, it may not keep working indefinitely. . . .

[17]The real surprises come when we find *heavy* particles that *don't* decay into lighter ones [or at least not right away]; this always means some hitherto unsuspected **conserved property** like "strangeness" or "charm" — but now I really *am* getting too far ahead!

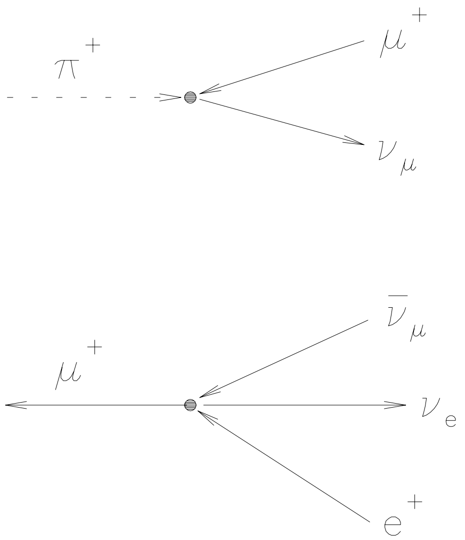

Figure 7: Top: Feynman diagram for decay of a π^+ meson [mass = 139 MeV/c^2] into a positive **muon** (μ^+) [mass = 106.7 MeV/c^2] and a [massless] muon **neutrino** (ν_μ). Bottom: Feynman diagram for decay of a μ^+ into a muon antineutrino ($\bar{\nu}_\mu$), a positron (e^+) and an electron neutrino (ν_e). These are the reactions I use in almost all of my research.

$E{=}mc^2$.3 Lorentz Invariants

In the previous handout we encountered the notion of *4-vectors*, the prototype of which is the **space-time** vector, $x_\mu \equiv \{ct, \vec{\mathbf{x}}\} \equiv \{x_0, x_1, x_2, x_3\}$, where the "zeroth component" x_0 is *time* multiplied by the speed of light: $x_0 \equiv ct$ and the remaining three components are the three ordinary spatial coordinates. [The notation is new but the idea is the same.] In general we assume that a vector with *Greek indices* (like x_μ) is a *4-vector*, while a vector with *Roman indices* (like x_i) is an ordinary spatial 3-vector.

It can be shown[18] that the **inner** or **scalar product** of any two *4-vectors* has the agreeable property of being a **Lorentz invariant** — *i.e.*, it is unchanged by a **Lorentz transformation** — *i.e.*, it has the *same value for all observers*. This comes in *very* handy in the confusing world of Relativity! We write the **scalar product** of two *4-vectors* as follows:

$$a_\mu b^\mu \;\equiv\; \sum_{\mu=0}^{3} a_\mu b^\mu \;=\; a_0 b_0 \;-\; \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \;=\; a_0 b_0 \;-\; (a_1 b_1 + a_2 b_2 + a_3 b_3) \tag{10}$$

where the first equivalence expresses the **Einstein summation convention** — we automatically *sum over repeated indices*. Note the − sign! It is part of the definition of the "metric" of space and time, just like the **Pythagorean theorem** defines the "metric" of flat 3-space in Euclidean geometry.

Our first **Lorentz invariant** was the **proper time** τ of an event, which is just the

[18]Don't you hate that phrase? Me too, but for now I have to resort to it.

square root of the scalar product of the *space-time 4-vector* with *itself*:

$$\tau \;=\; \sqrt{x_\mu x^\mu} \;=\; \sqrt{c^2t^2 - \vec{\mathbf{x}}\cdot\vec{\mathbf{x}}} \tag{11}$$

We now encounter our second *4-vector*, and probably the last one we need for our purposes here; the **energy-momentum** *4-vector*,

$$p_\mu \;\equiv\; \{\tfrac{E}{c}, \vec{\mathbf{p}}\} \;\equiv\; \{\tfrac{E}{c}, p_x, p_y, p_z\} \tag{12}$$

where $cp_0 \equiv E = \gamma mc^2$ is the **total relativistic energy** and $\vec{\mathbf{p}}$ is the usual **momentum** 3-vector of some object in whose kinematics we are interested. [Check for yourself that all the components of this vector have the *same units*, as required.] If we take the scalar product of p_μ with itself, we get a new **Lorentz invariant**:

$$p^\mu p_\mu \;\equiv\; \frac{E^2}{c^2} - \vec{\mathbf{p}}\cdot\vec{\mathbf{p}} \;=\; \frac{E^2}{c^2} \;-\; p^2 \tag{13}$$

where $p^2 \equiv \vec{\mathbf{p}}\cdot\vec{\mathbf{p}}$ is the square of the magnitude of the ordinary 3-vector momentum.

It turns out[19] that the constant value of this particular **Lorentz invariant** is just the c^2 times the *square of the* **rest mass** of the object whose momentum we are scrutinizing: $\frac{E^2}{c^2} - p^2 = m^2c^2$ or $E^2 - p^2c^2 = m^2c^4$. As a result, we can write

$$E^2 \;=\; p^2c^2 \;+\; m^2c^4 \tag{14}$$

which is a *very* useful formula relating the **energy** E, the **rest mass** m and the **momentum** p of a relativistic body.

Although there are lots of other **Lorentz invariants** we can define by taking the scalar products of *4-vectors*, these two will suffice for my purposes; you may forget this derivation entirely if you so choose, but I will need Eq. (14) for future reference.

E=mc^2.3.1 The Mass of Light

Allow me to hearken momentarily back to Newton's picture of light as *particles*.[20] Actually the following analysis pertains to *any* particles *whose rest mass is zero*. If $m = 0$ then Eq.(3) is absurd, except in the rather useless sense that we may let γ become infinite. On the other hand, Eq.(14) works fine if $m = 0$. Then we just have

$$E \;=\; pc \tag{15}$$

— that is, the **energy** and **momentum** of a *massless* particle differ only by a factor of c, its speed of propagation. Although we cannot define γ because the massless particle *always*

[19]Ouch! There's another one.

[20]This is also a preview of topics to come; as we shall see later, Newton was quite right! Light *does* come in well-defined *quanta* known as **photons**, particles of zero rest mass that always propagate at the speed of you-know-what!

moves at c relative to *any* observer [this was, after all, one of the original postulates of the $\mathcal{STR}$], we can talk about its **effective mass**, which is the same as its **kinetic energy** divided by c^2.

Thus, even though light has no **rest mass** (because it can never be at rest!), it *does* have an effective mass which (it turns out) has all the properties one expects from **mass** — in particular, it has *weight* in a gravitational field [photons can "fall"] and exerts a gravitational attraction of its own on other masses. The classic *Gedankenexperiment* on this topic is one in which the *net mass* of a closed box with mirrored sides *increases* if it is filled with *light* bouncing back and forth off the mirrors!

Is that weird, or what?