SOME HANDY MATH

The following useful mathematical tools are not provided in the textbook as far as I know.

M.1 Delta Functions

The so-called DIRAC DELTA FUNCTION $\delta(x - x_\circ)$ is defined by its properties:

- It is normalized: $\displaystyle\int_{-\infty}^{+\infty} \delta(x - x_\circ)\, dx = 1.$
- It is zero everywhere but at $x = x_\circ$. (This makes mathematicians very uncomfotable; they prefer to call $\delta(x - x_\circ)$ a "distribution" rather than a "function.")
- (most important) $\displaystyle\int_{-\infty}^{+\infty} f(x)\, \delta(x - x_\circ)\, dx = f(x_\circ).$

It has a number of handy <u>representations</u>:

- Gaussian representation: $\displaystyle\delta(x - x_\circ) = \frac{1}{\sqrt{2\pi}} \cdot \lim_{\sigma\to 0} \left\{ \frac{1}{\sigma} \cdot \exp\left[-\frac{(x - x_\circ)^2}{2\sigma^2} \right] \right\}.$
- Lorentzian representation: $\displaystyle\delta(x - x_\circ) = \frac{1}{\pi} \cdot \lim_{\gamma\to 0} \left\{ \frac{\gamma}{(x - x_\circ)^2 + \gamma^2} \right\}.$
- Kronecker representation (square "hill"):

$$\delta(x - x_\circ) = \left\{ \begin{array}{ll} \frac{1}{\gamma} & \text{for} \quad -\frac{\gamma}{2} < (x - x_\circ) < \frac{\gamma}{2} \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{as} \quad \gamma \to 0.$$

- Fourier Integral representation: $\displaystyle\delta(x - x_\circ) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{ik(x - x_\circ)}\, dk.$

M.2 Nasty Integrals

- Error Function: $\displaystyle\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2}\, dx \quad \underset{y\to\infty}{\longrightarrow} \ 1$
- $\displaystyle\int_0^\infty \frac{x^3\, dx}{e^x - 1} = \frac{\pi^4}{15}$

M.3 Central Limit Theorem

This theorem describes the behaviour of a *sum of random variables* as a random variable:

If $y \equiv \sum_{i=1}^{N} x_i$ for $N \gg 1$, where each x_i is a random sample from a distribution $\mathcal{D}(x)$ with mean $\langle x \rangle$ and variance $\delta_x^2 \equiv \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$, then

$$\mathcal{D}(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\delta_y} \cdot \exp\left[-\frac{(y - \langle y \rangle)^2}{2\delta_y^2} \right]$$

where $\langle y \rangle = N \langle x \rangle$ (*"The mean of the sum is the sum of the means"*)
and $\delta_y^2 = N \delta_x^2$ (*"The variance of the sum is the sum of the variances"*),
regardless of the shape of $\mathcal{D}(x)$ [as long as $\langle x \rangle$ and δ_x^2 are finite and well-defined].

Note especially that $\mathcal{D}(y)$ is *always* a *normal (Gaussian)* distribution!

I have used an unconventional notation here (δ instead of σ for the variance) because K&K use σ for the *entropy* and I don't want to create more notational ambiguity than necessary.

M.4 Elementary Expansions, Limits & Approximations

Assume throughout that $\delta \ll 1$.

- Power Law: $(1+\delta)^p = 1 + p\,\delta + (p-1)\,\dfrac{\delta^2}{2} + \cdots \approx 1 + p\,\delta$

  Especially useful: $p = -1$ for which $\dfrac{1}{1 \pm \delta} \approx 1 \mp \delta$

- Logarithm: $\log(1+\delta) = \delta - \dfrac{\delta^2}{2} + \cdots \approx \delta$

- Exponential: $e^\delta = 1 + \delta + \dfrac{\delta^2}{2} + \cdots \approx 1 + \delta$

  Especially useful: $e^{-\delta^2} \approx 1 - \delta^2$

  Note also that $\lim_{x \to 0} e^{\pm x} = 1$ while

  $$\lim_{x \to \infty} e^{+x} \longrightarrow \infty \quad \text{fast!} \qquad \text{and} \qquad \lim_{x \to \infty} e^{-x} \longrightarrow 0 \quad \text{fast!}$$

- Stirling's approximation:

  $$N! \simeq \sqrt{2\pi N} \cdot N^N \cdot e^{-N} \qquad \left[\cdot \exp\left\{ \tfrac{1}{12N} + \mathcal{O}\left(\tfrac{1}{N^2}\right) \right\} \right] \qquad \text{or}$$

  $$\log N! \simeq N \log N \; - \; N \qquad \left[+ \tfrac{1}{2}\log(2\pi N) + \tfrac{1}{12N} + \mathcal{O}\left(\tfrac{1}{N^2}\right) \right]$$